6 Number System

6.1 Decimals

The use of numbers involving decimal points is very important. Recall that:

$$\frac{1}{10} = 0.1 \qquad \text{"one tenth"}$$

$$\frac{2}{10} = 0.2 \qquad \text{"two tenths"}$$

$$\frac{1}{100} = 0.01 \qquad \text{"one hundredth"}$$

$$\frac{7}{100} = 0.07 \qquad \text{"seven hundredths"}$$

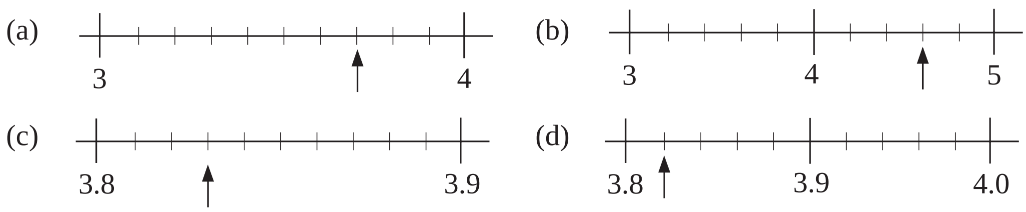
$$\frac{42}{100} = 0.42 \qquad \text{"4 tenths and 2 hundredths"}$$

$$\frac{82}{100} = 0.82 \qquad \text{"8 tenths and 2 hundredths"}$$

$$\frac{7}{1000} = 0.007 \qquad \text{"7 thousandths"}$$

Worked Example 1

Read the value indicated by each pointer.



Solution

(a) Each mark on the scale is 0.1 units apart, so the arrow points to 3.7.

(b) Each mark on the scale is 0.2 units apart, so the arrow points to 4.6.

(c) Each mark on the scale is 0.01, so the arrow points to 3.83.

(d) Each mark on the scale is 0.02 units apart, so the arrow points to 3.82.

Worked Example 2

Find

(a) $0.17 + 0.7$ (b) $0.624 + 0.41$ (c) $0.12 + 0.742$

Solution

(a) $\begin{array}{r} 0.17 \\ +\;\; 0.7 \\ \hline 0.87 \end{array}$ (b) $\begin{array}{r} 0.624 \\ +\;\; 0.41 \\ \hline 1.034 \end{array}$ (c) $\begin{array}{r} 0.12 \\ +\;\; 0.742 \\ \hline 0.862 \end{array}$

Note how the decimal points are lined up above each other.

Worked Example 3

A boy spent 48 p on football stickers, 33 p on sweets and 95 p on a comic. Find the total he spent in £s.

Solution

$$\begin{array}{r} 48 \\ 33 \\ +\;\; 95 \\ \hline 176 \end{array}$$

He spent 176 p or £1.76.

Exercises

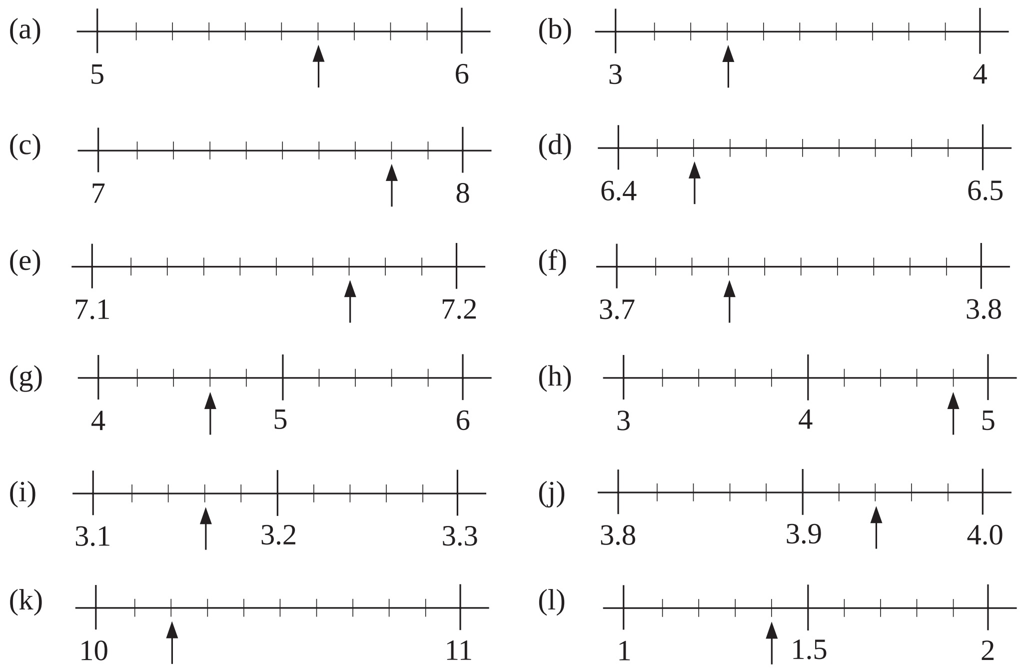
1. Write each of these as a decimal.

(a) $\frac{7}{10}$	(b) $\frac{8}{10}$	(c) $\frac{3}{10}$
(d) $\frac{5}{100}$	(e) $\frac{21}{100}$	(f) $\frac{42}{100}$
(g) $\frac{5}{1000}$	(h) $\frac{151}{1000}$	(i) $\frac{22}{1000}$
(j) $\frac{8}{100}$	(k) $\frac{13}{100}$	(l) $\frac{16}{1000}$
(m) $\frac{5}{10}$	(n) $\frac{4}{100}$	(o) $\frac{321}{1000}$

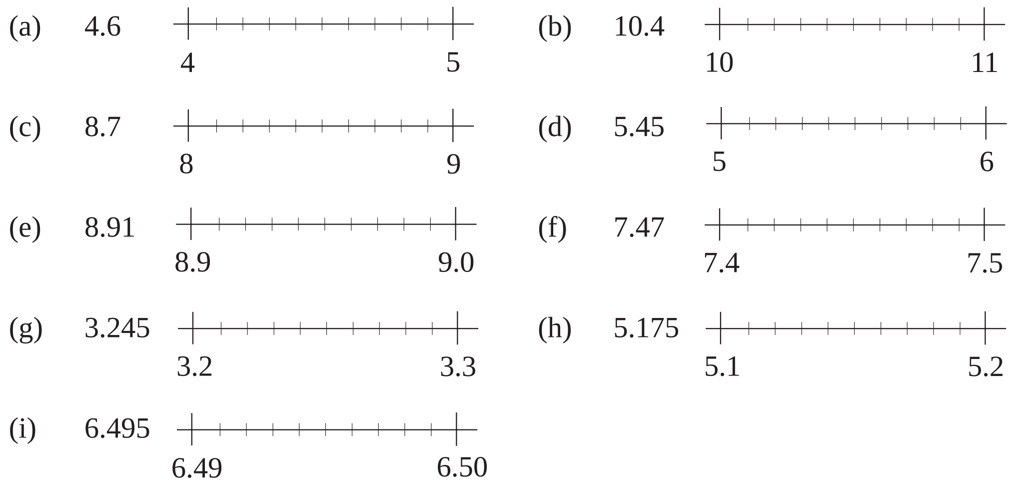
2. Write each of these as a fraction.

(a) 0.4	(b) 0.3	(c) 0.04
(d) 0.32	(e) 0.45	(f) 0.06
(g) 0.08	(h) 0.14	(i) 0.008
(j) 0.147	(k) 0.036	(l) 0.04
(m) 0.1	(n) 0.009	(o) 0.107

3. Read the value indicated by each pointer.



4. On a copy of the scale, mark as accurately as possible the given value.



5. Find.

(a)	$0.7 + 0.81 =$	(b)	$0.004 + 0.42 =$	(c)	$0.1 + 0.182 =$
(d)	$0.863 - 0.024 =$	(e)	$0.802 + 0.3 =$	(f)	$0.321 - 0.04 =$
(g)	$0.86 - 0.002 =$	(h)	$0.85 - 0.112 =$	(i)	$0.386 - 0.014 =$
(j)	$8.67 + 3.2 =$	(k)	$8.571 + 3.72 =$	(l)	$4.8 + 12.68 =$
(m)	$18.2 - 9.47 =$	(n)	$28.6 + 102.05 =$	(o)	$82.01 - 32.004 =$

Information

The sides of the Great Pyramid of Giza in Egypt are about 230.5 m long. Although it was built thousands of years ago by thousands of slaves, the lengths of its sides vary by no more than 11.5 cm!

6. In each number below, does the 5 represent

5 tenths, 5 hundredths or 5 thousandths?

(a) 0.152 (b) 0.522 (c) 0.05

(d) 3.572 (e) 1.475 (f) 3.115

7. The table below shows how five children spent pocket money in one week.

Child	**Sweets**	**Stickers**	**Toys**	**Magazines/Comics**
Hester	84 p	24 p	70 p	80 p
Katie	25 p	–	80 p	49 p
Relah	16 p	48 p	99 p	65 p
Hien	86 p	48 p	42 p	99 p

(a) Find the total that each child spent in 2 weeks.

(b) Find the amount that the four children spent together in one week.

(c) How much did they spend on sweets in one week?

8. Convert the following amounts in pence to £s.

(a) 328 p (b) 152 p (c) 842 p

(d) 1121 p (e) 48 p (f) 127 p

(g) 64 p (h) 32 011 p (i) 8421 p.

9. Mr Krishnan buys five 20 p stamps and three 26 p stamps.

(a) How much does he spend in £'s?

(b) How much change would he get from a £5 note?

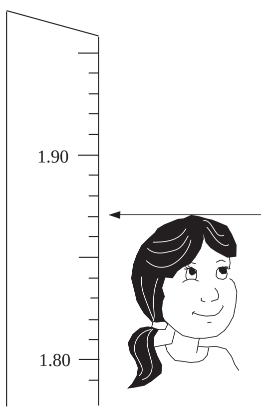
10. The cost of travelling by bus from Mrs Barnes' home to the city centre is £1.10 for adults and 65 p for children.

(a) Find the cost if Mrs Barnes goes to the city centre with her two children.

(b) Find the cost if Mr Barnes also goes.

11. Jackie measures her height.

How tall is she?

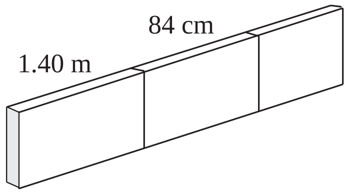


(LON)

12. Chippy the carpenter marks a 3 metre length of wood into three pieces.

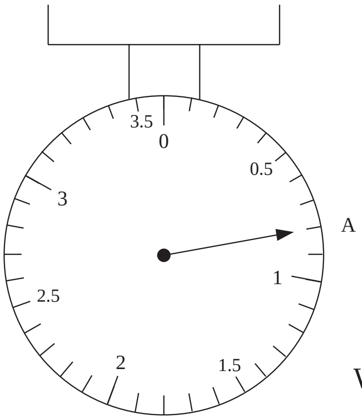
One piece is 1.40 metres long.
Another piece is 84 centimetres long.

How long is the third piece of wood?



(LON)

13.



These kitchen scales weigh in kilograms.

Write down the weight when the pointer is at A.

(MEG)

14. John goes into a post-office to buy some 19 p stamps.

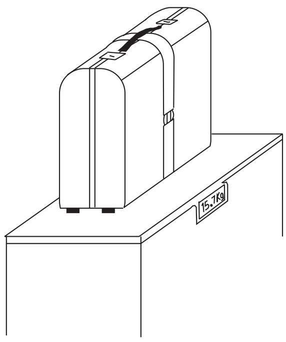
(a) What is the greatest number of stamps he can buy for £1?

John pays for the stamps with a £1 coin.

(b) How much change should John get?

(LON)

15. Ann flies from Manchester to Hong Kong.



At Manchester Airport her case is weighed and the scales show 15.7 kg.

In Hong Kong she buys four presents for her family.

They weigh:

4 kg, 2.50 kg, 0.75 kg, 3.60 kg.

(a) What is the total weight of these presents in kilograms?

(b) Ann puts the presents in her case when she packs it to fly home.
What does it weigh now?

(c) If her case now weighs more than 20 kg, there is an extra charge.
She has to pay 15 dollars for every kg or part kg over 20 kg.

How much does Ann have to pay?

(NEAB)

6.2 Multiplying and Dividing With Decimals

When multiplying or dividing by 10, 100, 1000, etc. the decimal point can simply be moved to the left or the right. When numbers such as 20, 200 or 300 are involved, the numbers can be multiplied by 2 or 3 and then the decimal point can be moved the correct number of places.

Worked Example 1

Find

(a) 362×100 (b) 4.73×10 (c) $576 \div 10$ (d) $4.2 \div 1000$

Solution

(a) To multiply by 100 move the decimal point 2 places to the right.

To do this it is necessary to add two zeros to the number. So

$$\begin{aligned} 362 \times 100 &= 362.00 \times 100 \\ &= 36\ 200. \end{aligned}$$

(b) To multiply by 10, move the decimal point one place to the right. So

$$4.73 \times 10 = 47.3.$$

(c) To divide by 10, move the decimal point one place to the left. So

$$576 \div 10 = 57.6.$$

(d) To divide by 1000 move the decimal point three places to the left. To do this it is necessary to put some extra zeros in front of the number.

$$4.2 \div 1000 = 0.0042.$$

Worked Example 2

Find:

(a) 3.4×20 (b) $\dfrac{14.8}{20}$ (c) $\dfrac{42}{0.7}$

Solution

(a) First multiply the 3.4 by 2 to give 6.8. Then multiply the 6.8 by 10 to give 68; so

$$\begin{aligned} 3.4 \times 20 &= 3.4 \times 2 \times 10 \\ &= 6.8 \times 10 \\ &= 68. \end{aligned}$$

(b) First divide 14.8 by 2 to give 7.4. Then divide by 10 to give 0.74.

$$\begin{aligned} \frac{14.8}{20} &= \frac{7.4}{10} \\ &= 0.74. \end{aligned}$$

(c) First multiply both numbers by 10 so that the 0.7 becomes a 7. This will make the calculation easier. Then divide 420 by 7 to give 60.

$$\frac{42}{0.7} = \frac{420}{7}$$
$$= 60.$$

Exercise

1. Find.

(a)	4.74×10	(b)	6.32×100	(c)	$41.6 \div 10$
(d)	12.74×100	(e)	$16.58 \div 100$	(f)	$32.4 \div 10$
(g)	6.3×100	(h)	4.7×1000	(i)	$3.2 \times 10\ 000$
(j)	47×1000	(k)	$6.8 \div 1000$	(l)	$82 \div 100$
(m)	$192 \div 1000$	(n)	$14 \div 1000$	(o)	0.18×1000

2. Find.

(a)	1.8×20	(b)	4.7×300	(c)	15×700
(d)	66×2000	(e)	15×400	(f)	1.3×8000
(g)	$66 \div 20$	(h)	$74 \div 200$	(i)	$21 \div 3000$
(j)	$35 \div 5000$	(k)	$3.42 \div 20$	(l)	$52 \div 400$
(m)	18.1×600	(n)	47.2×500	(o)	$4.95 \div 50$
(p)	3×0.02	(q)	15×0.04	(r)	5×0.0007

3. Find.

(a)	$\frac{16}{0.4}$	(b)	$\frac{500}{0.2}$	(c)	$\frac{64}{0.8}$
(d)	$\frac{24}{0.04}$	(e)	$\frac{264}{0.02}$	(f)	$\frac{465}{0.15}$
(g)	$\frac{156}{0.03}$	(h)	$\frac{48}{0.012}$	(i)	$\frac{56}{0.08}$

4. A factory produces washers which it sells at 1.2 pence each.

(a) Find the income in pence from the sale of:

(i) 300 washers (ii) 50000 washers (iii) 4000 washers.

(b) Convert your answers to (a) from pence to pounds.

(c) £3600 was paid for a batch of washers. How many washers were in this batch?

5. A company made a large profit one year and decided to give a bonus to each department. The bonus was divided equally among all the staff in each department.

Department	**Total Bonus**	**Number of staff**
Production	£12 487	100
Sales	£8 260	20
Delivery	£5 350	50
Finance	£4 896	40

Find the bonus that would be paid to staff in each department.

6. A snail moves at a speed of 0.008 miles per hour.

(a) How far would the snail travel in 1.5 hours?

(b) How long would it take the snail to travel:

(i) 40 miles (ii) 0.72 miles?

7. The cost of making a chocolate bar is 2.7 pence.

(a) What is the cost of producing:

(i) 4000 (ii) 17 000 (iii) 30 000 chocolate bars?

(b) A consultant says that he can reduce the production costs by 0.4 pence per bar. How much would this save on the production of:

(i) 5 000 (ii) 22 000 (iii) 30 000 chocolate bars?

8. A new pop group are trying to produce their first CD.

(a) They are told that it will cost £1.20 to make each CD. If they can afford to spend £1800 on producing the CDs, how many can they make?

(b) One of the group find another CD manufacturer who will manufacture the CDs for 90 pence each. How many more can they produce at this price?

9. It is established that a lorry can carry 64000 cans of soft drinks. Each can contains 0.33 litres of drink.

Find the total volume of the drink carried by the lorry.

10. For a major sporting event, a stadium is expected to hold its limit of 70 000 spectators.

(a) How much money is taken in ticket sales if the price of the tickets were:

(i) £5 (ii) £8 (iii) £11?

(b) If £432 000 is taken in ticket sales when the ticket price is £6, how many spectators will not be able to get into the ground?

11. (a) $\boxed{900 \times 0.6}$

Work out the answer to this sum in your head. *Do not use a calculator.*

Explain clearly the method you used.

(b) $\boxed{40 \div 0.8}$

Work out the answer to this sum in your head. *Do not use a calculator.*

Explain clearly the method you used.

(NEAB)

12. (a) Multiply 65 by 100.

(b) Write the number one thousand and thirty seven in figures.

(c) Add your answer for part (b) to your answer for part (a).

(MEG)

13. Fill in the missing numbers.

(a) 7_ × 100 = _2_0 (b) _0 × 30 = _80_

(NEAB)

14. (a) Write down the value of:

(i) 2×9 (ii) 9×9.

(b) Use your answers in part (a) to calculate the value of 29×9, showing your working in full.

(MEG)

6.3 Fractions and Decimals

Some fractions can be written as decimals with a fixed number of decimal places, for example:

$$\frac{1}{4} = 0.25$$

These are called *terminating* decimals. Others have an infinite number of decimal places, for example:

$$\frac{1}{3} = 0.333\,333\ldots$$

Numbers that contain an infinite number of decimal places are usualy rounded to a specified number of significant figures or decimal places.

Worked Example 1

Round each number in the list below to:

(i) 3 significant figures (ii) 3 decimal places.

(a) 4 732.165 (b) 4.736 1 (c) 417.923 5

(d) 0.056 234 (e) 0.004 721

Solution

(a) (i) $4\,732.165 = 4\,730$ to 3 significant figures.
Note that only the first 3 figures are considered.

(ii) $4\,732.165 = 4\,732.165$ to 3 decimal places. There is no charge as there are exactly 3 figures behind the decimal point.

(b) (i) $4.736\,1 = 4.74$ to 3 significant figures. The first three figures are considered and the 3 is rounded up to a 4, because it is followed by a 6.

(ii) $4.736\,1 = 4.736$ to 3 decimal places. The 6 is not rounded up because it is followed by a 1.

(c) (i) $417.923\,5 = 418$ to 3 significant figures. The first 3 figures are used and the 7 is rounded up to 8 because it is followed by a 9.

(ii) $417.923\,5 = 417.924$ to 3 decimal places. There are three figures behind the decimal point and the 3 is rounded up to a 4 because it is followed by a 5.

(d) (i) $0.056\,234 = 0.056\,2$ to 3 significant figures. Note that the zeros at the start of this number are not counted.

(ii) $0.056\,234 = 0.056$ to 3 decimal places.

(e) (i) $0.004\,721 = 0.004\,72$ to 3 significant figures. Note that the zeros in front of the 4 are not counted.

(ii) $0.004\,721 = 0.005$ to 3 decimal places. The 4 is rounded up to a 5 because it is followed by a 7.

Worked Example 2

Convert each of the following fractions to decimals,

(a) $\frac{1}{4}$ (b) $\frac{2}{3}$ (c) $\frac{4}{5}$ (d) $\frac{3}{7}$

Solution

In each case the bottom number should be divided into the top number. This will require long division.

(a) To convert $\frac{1}{4}$, divide 4 into 1.

$$\begin{array}{r} 0.25 \\ 4\,\overline{)\,1.00} \\ 8 \\ \hline 20 \\ 20 \\ \hline 0 \end{array}$$

So $\frac{1}{4} = 0.25$.

(b) To convert $\frac{2}{3}$, divide 3 into 2.

$$
\begin{array}{r|l}
 & 0.666\ldots \\
\hline
3 & 2.000 \\
 & 1\ 8 \\
\hline
 & \ \ 20 \\
 & \ \ 18 \\
\hline
 & \ \ \ \ 2
\end{array}
$$

So $\frac{2}{3} = 0.666\ 6\ldots = 0.667$ to 3 decimal places.

(c) To convert $\frac{4}{5}$, divide 5 into 4.

$$
\begin{array}{r|l}
 & 0.8 \\
\hline
5 & 4.0 \\
 & 4\ 0 \\
\hline
 & \ \ \ 0
\end{array}
$$

So $\frac{4}{5}$ is exactly 0.8.

(d) To convert $\frac{3}{7}$ into a decimal divide 7 into 3.

$$
\begin{array}{r|l}
 & 0.42857\ \ldots \\
\hline
7 & 3.000000 \\
 & 2\ 8 \\
\hline
 & \ \ 20 \\
 & \ \ 14 \\
\hline
 & \ \ \ \ 60 \\
 & \ \ \ \ 56 \\
\hline
 & \ \ \ \ \ \ 40 \\
 & \ \ \ \ \ \ 35 \\
\hline
 & \ \ \ \ \ \ \ \ 50 \\
 & \ \ \ \ \ \ \ \ 49 \\
\hline
 & \ \ \ \ \ \ \ \ \ \ 1
\end{array}
$$

There will be an infinite number of decimal places in this case, but

$$\frac{3}{7} = 0.428\ 6$$

correct to 4 decimal places.

Just for Fun

Without moving 6 adjacent numbers of the face of a clock, rearrange the other six so that the sum of every pair of adjacent numbers is a prime number.

Exercises

1. Write each of the following numbers correct to:

   (i) 2 decimal places (ii) 2 significant figures.

(a)	18.643	(b)	1 024.837	(c)	16.04
(d)	181.435	(e)	16.824	(f)	0.083 741
(g)	0.009 562	(h)	4.837 5	(i)	3.864 9

2. Write the number 48 637.012 45 correct to

(a)	3 significant figures	(b)	2 decimal places
(c)	4 decimal places	(d)	4 significant figures
(e)	3 decimal places	(f)	2 significant figures.

3. Write each number correct to the number of decimal places or significant figures specified.

(a)	0.00472 (2 s.f.)	(b)	48.234 (3 s.f.)	(c)	15.83 (1 s.f..)
(d)	4.862 (2 d.p.)	(e)	18.415 (2 d.p.)	(f)	21.804 (2 d.p.)
(g)	14862 (2 s.f.)	(h)	0.00463 (3 d.p.)	(i)	0.004178 (3 s.f.)
(j)	15682 (3 s.f.)	(k)	54631 (2 s.f.)	(l)	31.432 (3 s.f.)
(m)	14.176 (4 s.f.)	(n)	0.815 (2 s.f.)	(o)	1.84149 (3d.p.)
(p)	15.013 (3 s.f.)	(q)	14.1704 (3 d.p.)	(r)	201.04 (3 s.f.)

4. The number of spectators that enter a football ground for a big match is 44 851.

   (a) Write this number correct to 1, 2, 3 and 4 significant figures.

   (b) Which of your answers to (a) makes the number of spectators appear

   (i) the largest (ii) the smallest?

5. Each of the fractions below can be written as a terminating decimal. Write each fraction as a decimal.

(a)	$\frac{1}{2}$	(b)	$\frac{3}{4}$	(c)	$\frac{2}{5}$
(d)	$\frac{3}{5}$	(e)	$\frac{1}{8}$	(f)	$\frac{5}{8}$
(g)	$\frac{3}{8}$	(h)	$\frac{7}{8}$	(i)	$\frac{1}{5}$

Information

Blaise Pascal (1623–1662) invented and made the first calculating machine at the age of 18 years.

6. Write each of he following fractions as a decimal correct to 4 decimal places.

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{4}{7}$

(d) $\frac{1}{7}$ (e) $\frac{5}{7}$ (f) $\frac{5}{6}$

7. (a) Write $\frac{1}{9}$, $\frac{2}{9}$, $\frac{4}{9}$ and $\frac{5}{9}$ as decimals correct to 5 decimal places.

(b) Describe any patterns that you notice in these decimals before they are rounded.

(c) How would you expect $\frac{7}{9}$ and $\frac{8}{9}$ to be written as decimals?

Check your answers.

8. (a) Write $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$ and $\frac{4}{11}$ as decimals correct to 5 decimal places.

(b) By looking at any patterns that you observe, write down

$$\frac{5}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{9}{11} \text{ and } \frac{10}{11}$$

as decimals.

(c) Check your answers for $\frac{7}{11}$ and $\frac{10}{11}$ by division.

9. Write down two different numbers that are the same when rounded to:

(a) 2 decimal places and 2 significant figures,

(b) 3 decimal places and 5 significant figures,

(c) 1 decimal place and 8 significant figures,

(d) 4 decimal places and 2 significant figures.

10. (a) Change $\frac{4}{5}$ to a decimal.

(b) Write these numbers in order of size. Start with the smallest.

$$0.805, \quad 0.85, \quad \frac{4}{5}, \quad 0.096.$$

(SEG)

11. P____

(a) Mark with an X a point approximately $\frac{1}{3}$ of the way along the line from P.

(b) Mark with a Z a point approximately 0.75 of the way along the line from P.

(LON)

6.4 Long Multiplication and Division

This section revises long multiplication and long division.

Worked Example 1

Find

(a) 127×24 (b) 146×137

Solution

(a)

$$
\begin{array}{rrl}
 & 1\,2\,7 & \\
 & \times \quad 2\,4 & \\
\hline
 & 5\,0\,8 & \leftarrow 127 \times 4 \\
127 \times 2 \rightarrow & 2\,5\,4\,0 & \leftarrow \text{ Insert } 0 \\
\hline
 & 3\,0\,4\,8 &
\end{array}
$$

(a)

$$
\begin{array}{rrl}
 & 1\,4\,6 & \\
 & \times 1\,3\,7 & \\
\hline
 & 1\,0\,2\,2 & \leftarrow 146 \times 7 \\
146 \times 3 \rightarrow & 4\,3\,8\,0 & \leftarrow \text{ Insert } 0 \\
146 \times 1 \rightarrow & 1\,4\,6\,0\,0 & \leftarrow \text{ Insert } 00 \\
\hline
 & 2\,0\,0\,0\,2 &
\end{array}
$$

Worked Example 2

Find

(a) $1675 \div 5$ (b) $312 \div 13$

Solution

(a)

$$
\begin{array}{r|l l}
 & 3\,3\,5 & \\
\hline
5 & 1\,6\,7\,5 & \\
 & 1\,5 & \leftarrow 3 \times 5 \\
 & 1\,7 & \\
 & 1\,5 & \leftarrow 3 \times 5 \\
 & 2\,5 & \\
 & 2\,5 & \leftarrow 5 \times 5 \\
 & 0 &
\end{array}
$$

(b)

$$
\begin{array}{r|l l}
 & 2\,4 & \\
\hline
13 & 3\,1\,2 & \\
 & 2\,6 & \leftarrow 2 \times 13 \\
 & 5\,2 & \\
 & 5\,2 & \leftarrow 4 \times 13 \\
 & 0 &
\end{array}
$$

Exercises

You should not use a calculator for these questions.

1.

(a)	$15 \times 23 =$	(b)	$18 \times 38 =$	(c)	$19 \times 27 =$
(d)	$64 \times 142 =$	(e)	$28 \times 261 =$	(f)	$48 \times 321 =$
(g)	$52 \times 49 =$	(h)	$128 \times 15 =$	(i)	$324 \times 72 =$
(j)	$84 \times 121 =$	(k)	$56 \times 42 =$	(l)	$38 \times 147 =$
(m)	$212 \times 416 =$	(n)	$58 \times 2312 =$	(o)	$4718 \times 12 =$

2.

(a)	$760 \div 5 =$	(b)	$762 \div 3 =$	(c)	$1038 \div 6 =$
(d)	$1004 \div 4 =$	(e)	$1356 \div 3 =$	(f)	$2996 \div 7 =$
(g)	$1476 \div 12 =$	(h)	$490 \div 14 =$	(i)	$228 \div 19 =$
(j)	$768 \div 24 =$	(k)	$432 \div 18 =$	(l)	$3366 \div 22 =$
(m)	$2144 \div 16 =$	(n)	$3638 \div 17 =$	(o)	$1573 \div 121 =$

3. Calculators are packed in boxes of 16. A shop receives 22 boxes of calculators and sells them for £6 each. How much money would the shop take if it sold all the calculators?

4. In a school every class has 28 pupils. If there are 25 classes in the school, what is the total number of pupils?

5. A sports supplier donates 156 footballs to a group of 12 schools. The balls are divided equally between the schools. How many footballs does each school get?

6. A delivery van contains 14 sacks of potatoes. Each sack has a mass of 25kg. Find the total mass of the potatoes.

7. A group of 6 people win £2000 in a competition. They share the prize out equally. Find the amount each person gets to the nearest penny.

8. The students who attended a sports training course are split into 16 groups. How many students are there in each group if:

   (a) 208 students attend (b) 112 students attend?

   A maximum of 15 students can be put in every group. What is the maximum number of students that can attend the course?

9. Cassette tapes are sold in packets of 15 which cost £11. John wants to buy 200 tapes. How much must be spent to get the 200 tapes?

10. A salesman travels an average of 742 miles per week. How far would he expect to travel in a year if he has:

    (a) 4 weeks holiday (b) 6 weeks holiday?

11. *Do not use your calculator in this question. Show all your working.*

    A school is planning a disco for 936 pupils. Each pupil will be given 1 can of drink. Cans of drink are sold in trays of 24.

    Work out how many trays of drink will be needed.

(LON)

12. Do not use your calculator in this question.

    (a) A travel company takes a party of people to a hockey match at Wembley. 17 coaches are used. Each coach has seats for 46 passengers. There are twelve empty seats altogether. How many people are in the party?

    Write down all your working to show you do not use a calculator.

    (b) 998 football supporters use another travel company to go to a football match at Wembley. Each coach has seats for 53 passengers.

    (i) How many coaches are needed?

    Write down all your working to show you do not use a calculator.

    (ii) How many empty seats are there?

(NEAB)

6.5 Estimating Answers

If you do a calculation such as

$$\frac{4.1721 \times 3.846}{18.21 + 5.73}$$

you need to use a calculator to find the answer. This section looks at ways of estimating the answers to calculations such as this.

Worked Example 1

Estimate the answers to each of the following problems.

(a) 18.42×3.76 (b) $\dfrac{47.932}{4.071}$ (c) $\dfrac{18.51 + 11.23}{3.0712}$

Solution

Estimates can be obtained by using each number correct to 1 or 2 significant figures.

(a) $$18.42 \times 3.76 \approx 20 \times 4 \approx 80$$

(b) $$\frac{47.932}{4.071} \approx \frac{48}{4} \approx 12$$

(c) $$\frac{18.51 + 11.23}{3.0712} \approx \frac{20 + 10}{3} \approx \frac{30}{3} \approx 10$$

Exercises

1. Write each of the following numbers correct to 1 significant figure.

(a)	47.316	(b)	18.45	(c)	27.65
(d)	9.632	(e)	15.01	(f)	149.32
(g)	62.84	(h)	0.176	(i)	0.039 4
(j)	1.964	(k)	21.87	(l)	1.849

2. Estimate the answers to the following problems:

(a)	6.74×8.31	(b)	4.35×12.46	(c)	236×4.321
(d)	16.67×3.21	(e)	5.92×105.3	(f)	16.78×32.51
(g)	$\dfrac{192.7}{17.35}$	(h)	$\dfrac{284}{37.2}$	(i)	$\dfrac{963}{51.8}$
(j)	$\dfrac{47.63}{0.4185}$	(k)	$\dfrac{36.72}{8.26}$	(l)	$\dfrac{17.24}{0.374}$

Now find the answer to each problem using a calculator, giving your answer to 4 significant figures. In each case compare your answers and estimates.

3. Estimate the answers to each of the following calculations.

(a)	$\dfrac{6.6 \times 9.5}{32.4}$	(b)	$\dfrac{0.32 \times 8.43}{6.21}$	(c)	$\dfrac{12.8 + 45.3}{17.3}$
(d)	$\dfrac{33.6 + 77.9}{15.72}$	(e)	$\dfrac{888 + 723}{38.4}$	(f)	$\dfrac{560 + 2.01}{29.47}$
(g)	$\dfrac{16.5 \times 3.82}{4.162}$	(h)	$\dfrac{82.4 + 91.9}{1.04 + 1.43}$	(i)	$\dfrac{82.6 \times 19.41}{0.024 \times 405}$

4. When cars leave a factory they are parked in a queue until they are delivered. The length of each car is 4.32 m. A queue contains 54 cars.
   (a) Estimate the length of the queue, if there are no gaps between the cars.
   (b) Find the length of the queue if there are no gaps between the cars.
   (c) If there is a gap of 0.57 m between each car, estimate the length and find the actual length.

5. A cross-country runner has an average speed of 6.43 m s^{-1} .
   (a) Estimate and find the distance run in 200 seconds, if he runs at his average speed.
   (b) Estimate and find, to 3 significant figures, the time it takes him to run 1473 m.

6. Drivers at a motor racing circuit complete practice laps in times of 130.21, 131.43 and 133.62 seconds. The length of the circuit is 5214 metres.

   (a) Estimate the average speed of the drivers.

   (b) Find their speeds correct to 2 decimal places.

7. A car travels 12.43 km on 1.12 litres of petrol.

   (a) Estimate and then calculate the distance that the car would travel on 1 litre of petrol.

   (b) Estimate the distances that the car would travel on 41.1 litres and 33.8 litres of petrol.

8. A factory produces 108 portable CD players every day. The cost of producing the CD players is made up of £4 125 for labour costs and £2 685 for parts.

   Estimate and then calculate:

   (a) the total cost of producing a CD player,

   (b) the cost of the parts to make a CD player,

   (c) the cost of the labour to make a CD player.

9. Carpet tiles are made so that they are square with sides of length 48 cm.

   Estimate and then calculate the number of tiles needed for rooms with sizes:

   (a) 6.41 m by 3.28 m (b) 3.84 m by 2.91 m (c) 4.29 m by 4.62 m.

10. (a) Write down the numbers you could use to get an approximate answer to

$$59 \times 32 .$$

   (b) Write down your approximate answer.

   (c) Using a calculator find the difference between your approximate answer and the exact answer.

(LON)

11. Flour costs 48 p per kilogram. Brett bought 205 kg and shared it equally among 14 people. He calculated that each person should pay £0.72.

   Without using a calculator, use a rough estimate to check whether this answer is about the right size.

   You must show all your working.

(SEG)

Investigation

A man died leaving behind 23 cows to his three children. His will stated that the eldest child should have half of the fortune, the second child should have one third and the youngest one eighth . The childrens could not decide how to divide up the cows without it being necessary to kill any of them.

A wise man came to the scene. He brought along his only cow and put it with the other 23 cows to give a total of 24 cows. He gave half of the 24 cows (12) to the eldest child, one third of the 24 cows (8) to the second child and one eighth of the 24 cows to the youngest child. He then took his own cow back. Can you discover the clue to this solution?

6.6 Using Brackets and Memory On a Calculator

By using the bracket and memory keys on a calculator it is possible to carry out tasks fairly quickly and easily.

Some of the standard memory keys which are found on a calculator are:

$\boxed{\text{Min}}$ Places the current number into the memory, replacing any previous number.

$\boxed{\text{MC}}$ Clears the memory.

$\boxed{\text{M+}}$ Adds the number displayed to the memory.

$\boxed{\text{MR}}$ Recalls the number that is currently in the memory.

Brackets can be used to tell the calculator the order in which to do calulations.

For example, to find:

$$\frac{3.62 + 4.78}{3.9 - 1.4},$$

use

$$\boxed{(}\ \boxed{3}\ \boxed{.}\ \boxed{6}\ \boxed{2}\ \boxed{+}\ \boxed{4}\ \boxed{.}\ \boxed{7}\ \boxed{8}\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ \boxed{3}\ \boxed{.}\ \boxed{9}\ \boxed{-}\ \boxed{1}\ \boxed{.}\ \boxed{4}\ \boxed{)}\ \boxed{=}$$

Worked Example 1

Find:

(a) $\dfrac{3}{3.2 + 1.8}$ (b) $\sqrt{\left(\dfrac{5.2 - 3.6}{4.7}\right)}$

Solution

(a) Use the brackets as shown below

$$\boxed{3}\ \boxed{\div}\ \boxed{(}\ \boxed{3}\ \boxed{.}\ \boxed{2}\ \boxed{+}\ \boxed{1}\ \boxed{.}\ \boxed{8}\ \boxed{)}\ \boxed{=}$$

to obtain 0.6.

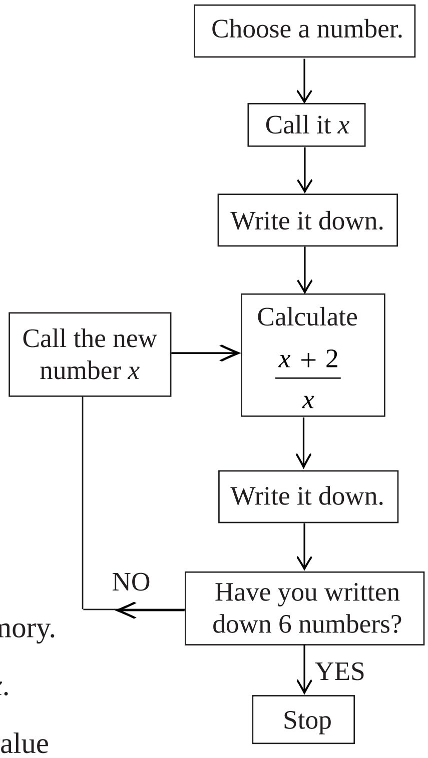
(b) Use brackets to enclose the top part of the fraction ,as shown below,

$$\boxed{(}\ \boxed{5}\ \boxed{.}\ \boxed{2}\ \boxed{-}\ \boxed{3}\ \boxed{.}\ \boxed{6}\ \boxed{)}\ \boxed{\div}\ \boxed{4}\ \boxed{.}\ \boxed{7}\ \boxed{=}\ \boxed{\sqrt{\ }}$$

to obtain 0.5835 correct to 4 decimal places.

Worked Example 2

Follow the instructions given in the flow chart for a student who chooses the number 20 as a starting point.



Solution

Starting with 20 leads to the calculation

$$\frac{20+2}{20} = 1.1$$

To perform the remaining calculations, follow the steps below.

1. Press Min to place the value displayed in the memory.
2. Press + 2 = which adds 2 to the value of x.
3. Press ÷ MR = which divides the displayed value by the number in the memory.
4. Go back to Step 1.

Worked Example 3

A factory produces plastic tanks in 4 different sizes. The table shows the orders placed one day.

Tank Size	Price	Number Ordered
Giant	£126	5
Large	£ 87	16
Medium	£ 56	44
Small	£ 33	31

Find the value of the orders, using the memory keys on your calculator.

Solution

1. First press MC to clear the memory.
2. For the *Giant* tanks, the value of the order is given by 126×5. Find this on your calculator and press the M+ key.
3. For the *Large* tanks, find 87×16 and press M+ again.
4. For the *Medium* tanks, find 56×44 and press M+ again.
5. For the *Small* tanks, find 33×31 and press M+ again.
6. Finally press MR to obtain the total, which is £5509.

Exercise

1. Carry out the following calculations, using the bracket keys on your calculator. *Give all answers to 3 significant figures.*

(a) $4 \times (8.1 + 16.2) =$ (b) $(5.6 - 3.2) \times 11.4 =$ (c) $\dfrac{15.6 + 3.2}{5.3} =$

(d) $\dfrac{19 + 24}{16} =$ (e) $\dfrac{33}{127 - 84} =$ (f) $\dfrac{19 + 61}{20 + 32} =$

(g) $\sqrt{\dfrac{4}{9 + 24}} =$ (h) $\dfrac{14.1 \times 2}{18 + 4} =$ (i) $\sqrt{\dfrac{16 + 22}{18 - 4}} =$

(j) $\left(\dfrac{8.2 + 4}{13 + 7}\right)^2 =$ (k) $\dfrac{3 + 4.9}{7.32 \times 18.4} =$ (l) $\left(\dfrac{4.7 - 3.2}{8 \times 0.22}\right)^2 =$

2. Work through the flow chart of *Worked Example 2*, starting with a number of your own choice.

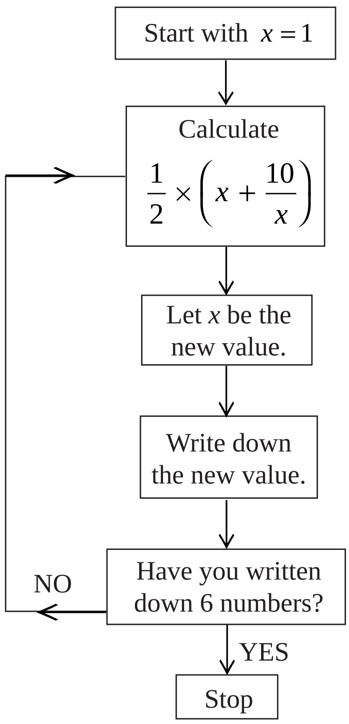
3. Find the mean of each set of numbers, using the brackets on your calculator.

(a) 15, 16, 17.5, 18, 20.

(b) 22, 21, 32, 28.

(c) 112, 114, 140, 130, 132, 126, 128, 110.

4. Use the flow chart shown in the diagram, giving final answers to 5 significant figures.



(a) Work through the flow chart as shown.

(b) Follow the flow chart but start with $x = 2$ instead of $x = 1$.

(c) How does this affect your final answer?

5. (a) Carry out the following calculation on your calculator inserting brackets where shown.

(i) $(24 \times 2) + (12 \times 4) + (3 \times 15) =$

$24 \times 2 \; + \; 12 \times 4 \; + \; 3 \times 15 \; =$

(ii) $(24 + 2) \times (15 + 3) =$

$24 + 2 \; \times \; 15 + 3 \; =$

(iii) $(24 \times 2) \div (5 \times 3) =$

$24 \times 2 \; \div \; 5 \times 3 \; =$

$24 \times 2 \div (5 \times 3) =$

(b) In each of the following decide which brackets, if any, could be missed out without changing the answer that would be obtained.

Check your answers with your calculator.

(i) $(3 \times 6) + (5 \times 51) + (15 \times 2) =$

(ii) $(3 + 6) \times (5 \times 2) =$

(iii) $(3 - 4) \times (8 - 2) =$

(iv) $(3 + 4) \div (5 \times 2) =$

(v) $(3 \times 4) \div (5 + 2) =$

(vi) $(3 \times 2) \div (4 \times 6) =$

6. The formula

$$A = 2\pi r(r + h)$$

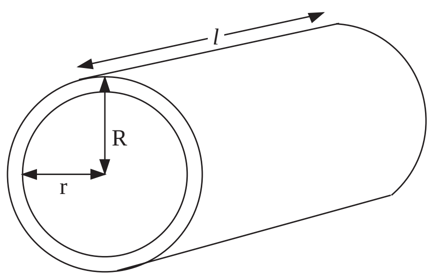
is used to calculate the surface area of a drinks can.

(a) Find A if $r = 6$ cm and $h = 10$ cm.

(b) Find A if $r = 3.7$ cm and $h = 7.4$ cm.

7. The volume of plastic used to make a pipe is given by the formula

$$V = \pi l\left(R^2 - r^2\right)$$



(a) Find V if

(i) $R = 25$ mm, $r = 20$ mm and $l = 3000$ mm,

(ii) $R = 3$ cm, $r = 2.4$ cm and $l = 500$ cm.

The formula can be rearranged as

$$l = \frac{V}{\pi\left(R^2 - r^2\right)}.$$

(b) Find l if:

(i) $V = 800 \text{ cm}^3$, $R = 5 \text{ cm}$ and $r = 4.5 \text{ cm}$.

(ii) $V = 100 \text{ cm}^3$, $R = 1 \text{ cm}$ and $r = 0.8 \text{ cm}$.

8. Find the value of f using the formula

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

if $u = 28.2$ and $v = 18.4$. Give your answer correct to 3 significant figures.

9. The acceleration due to gravity, g, on any planet can be found using the formula

$$g = \frac{G\,m}{d^2}$$

Find g if $G = 6.67 \times 10^{-11}$, $m = 7.4 \times 10^{30}$ and $d = 8.4 \times 10^{9}$.
Give your answer correct to 2 decimal places.

10. Use a calculator to find the value of

(a) $\dfrac{3.86 + 17.59}{5}$ (b) $\dfrac{9.76 + 1.87}{18.3 - 15.8}$

(c) $\dfrac{330}{1.2 \times 5.5}$ (d) $\dfrac{1}{\sqrt{(0.16)}}$

(NEAB)

11. Use your calculator to evalulate

(a) $(2.37 - 8.42)^2$ (b) $\sqrt{(2.37 - 8.42)^2 + 17.42}$

(MEG)

12. Some students are using calculators to work out four questions.

Question 1 $\dfrac{2.34 + 1.76}{3.22 + 1.85}$

Question 2 $\dfrac{2.34 + 1.76}{3.22} + 1.85$

Question 3 $2.34 + \dfrac{1.76}{3.22} + 1.85$

Question 4 $2.34 + \dfrac{1.76}{3.22 + 1.85}$

(a) Tom presses keys as follows:

$\boxed{2}\ \boxed{.}\ \boxed{3}\ \boxed{4}\ \boxed{+}\ \boxed{1}\ \boxed{.}\ \boxed{7}\ \boxed{6}\ \boxed{\div}\ \boxed{3}\ \boxed{.}\ \boxed{2}\ \boxed{2}\ \boxed{+}\ \boxed{1}\ \boxed{.}\ \boxed{8}\ \boxed{5}\ \boxed{=}$

For which of the four questions is this the correct method?

(b) Jayne presses keys as follows.

$\boxed{2}\ \boxed{.}\ \boxed{3}\ \boxed{4}\ \boxed{+}\ \boxed{1}\ \boxed{.}\ \boxed{7}\ \boxed{6}\ \boxed{=}\ \boxed{\div}\ \boxed{3}\ \boxed{.}\ \boxed{2}\ \boxed{2}\ \boxed{+}\ \boxed{1}\ \boxed{.}\ \boxed{8}\ \boxed{5}\ \boxed{=}$

For which of the four questions is this the correct method?

(SEG)

13. Trudie uses the formula

$$a = \frac{v - u}{t}.$$

She has to calculate the value of a when $v = 118.07$, $u = 17.76$ and $t = 4.8$.
Trudie estimates the value of a without using her calculator.

(a) (i) Write down numbers Trudie could use to estimate the value of a.

(ii) Write down the estimate these values would give for the value of a.

Trudie then uses her calculator to find the value of a.

(b) Here is the sequence of keys that she presses.

$\boxed{1}\ \boxed{1}\ \boxed{8}\ \boxed{.}\ \boxed{0}\ \boxed{7}\ \boxed{-}\ \boxed{1}\ \boxed{7}\ \boxed{.}\ \boxed{7}\ \boxed{6}\ \boxed{\div}\ \boxed{4}\ \boxed{.}\ \boxed{8}\ \boxed{=}$

This gives an answer of 114.37, which is not the correct answer.

Change the sequence above so that it will give the correct answer.

(LON)

14. Tony uses his calculator to work out

$$\frac{4.2 \times 86}{3.2 \times 0.47}.$$

He is told to do this in one sequence, writing down only the answer. He presses the keys as follows:

$\boxed{4}\ \boxed{.}\ \boxed{2}\ \boxed{\times}\ \boxed{8}\ \boxed{6}\ \boxed{\div}\ \boxed{3}\ \boxed{.}\ \boxed{2}\ \boxed{\times}\ \boxed{0}\ \boxed{.}\ \boxed{4}\ \boxed{7}\ \boxed{=}$

This gives him the wrong answer. Explain what is wrong with Tony's method.

(SEG)

Investigation

In country X, only 5 p and 8 p stamps are available. You have to post letters which cost 23 p, 27 p, 77 p and £19.51 respectively. Which of these amounts can you make exactly?

Make a comlpete list of the amounts between 1 p and 99 p which ***cannot*** *be made exactly.*

6.7 Upper and Lower Bounds

When measurements are made, they can only be obtained to a limited degree of accuracy. For example if the length of a line is given as 11mm, this means that it is 11mm to the nearest mm. In fact, if l is the length then it lies in the range

$$10.5 \le l < 11.5 .$$

Worked Example 1

For each length below state the range of values within which the length must be.

(a) 18 cm (b) 12.7 cm (c) 11.06 m

Solution

If l represents the given length then:

(a) $17.5 \le l < 18.5$ (b) $12.65 \le l < 12.75$ (c) $11.055 \le l < 11.065$

Note

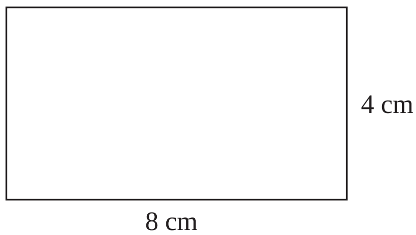
The *lower bound* is the smallest number which will round to the given number, while the *upper bound* is the smallest number which will *not* round to the given number.

Worked Example 2

If the length of the sides of the rectangle are given to the nearest cm, find:

(a) the minimum possible perimeter

(b) the range of possible areas.



Solution

First note that the length, l, of the rectangle is 8 cm to the nearest cm, so $7.5 \le l < 8.5$.
Similarly the width, w, of the rectangle lies in the range $3.5 \le w < 4.5$.

(a) To find the minimum possible perimeter, use the minimum possible length,

$$\begin{aligned}\text{Minimum perimeter} &= 7.5 + 3.5 + 7.5 + 3.5 \\ &= 22 \text{ cm}.\end{aligned}$$

(b) The range of values for the area can be calculated using the maximum and minimum lengths of each side.

$$\begin{aligned}\text{Minimum area} &= 3.5 \times 7.5 \\ &= 26.25.\end{aligned}$$

$$\begin{aligned}\text{Maximum area} &= 4.5 \times 8.5 \\ &= 38.25.\end{aligned}$$

So

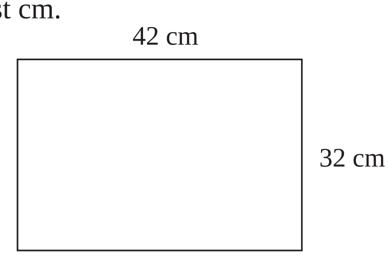
$$26.25 \le \text{area} < 38.25 .$$

Exercises

1. For each quantity below, state the range of values within which it must lie.

(a) $x = 4.7$	(b) $l = 42$	(c) $A = 15.62$
(d) $d = 16.2$	(e) $r = 11.68$	(f) $m = 14.24$
(g) $w = 218$	(h) $l = 15.20$	(i) $w = 5.00$
(j) $v = 20.0$	(k) $A = 18.09$	(l) $r = 31.451$

2. The lengths of the rectangle are given to the nearest cm.



(a) State the range of values within which the length, l, and the width, w, must lie.

(b) Find the range of values within which,

(i) the perimeter (ii) the area

of the rectangle must lie.

3. The radius of a circle is 12 mm to the nearest mm.

(a) State the range of possble values for the radius.

(b) Find the range of possible values for the:

(i) diameter (ii) circumference (iii) area.

4. The quantities x and y are given to 1 significant figure as $x = 20$ and $y = 40$. Find the minimum possible value of each expression below.

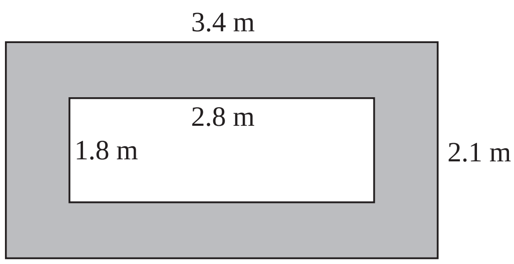
(a) $x + y$ (b) $y - x$ (c) xy (d) $\frac{x}{y}$ (e) $\frac{y}{x}$

5. A student runs 100 m in 12.8 seconds. Find the possible speeds of the student if:

(a) the distance is to the nearest metre and the time is to the nearest 0.1 s,

(b) the distance is to the nearest cm and the time is to the nearest 0.1 s.

6. A gardener is to pave the area around a flower bed. He marks out the position of the paved area and measures it to the nearest 0.1 m.



Find the minimum and maximum possible areas of the paved area.

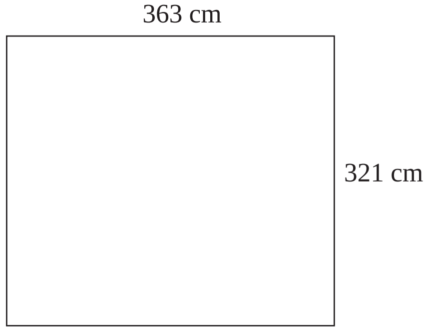
7. A carpet fitter measures the size of a room to the nearest cm.

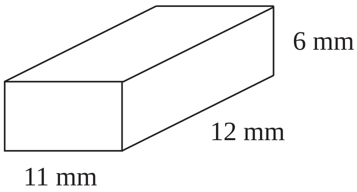
   The diagram shows this measurement.

   (a) Find the maximum and the minimum values for the area of the carpet used.

   The carpet fitter brings a carpet measured as 360 cm by 380 cm, to the nearest cm.

   (b) Find the maximum amount of carpet that will be wasted.



8. 

   The dimensions of a block of material are given to the nearest mm.

   Its mass is 4.2 grams, to the nearest 0.1 gram.

   Find a range of values within which the density of the material must lie.

9. Junctions 18, 19, 20 and 21 are on the same motorway. The distances between the junctions, to the nearest mile are:

Junction 18	to	Junction 19	22 miles
Junction 19	to	Junction 20	36 miles
Junction 20	to	Junction 21	12 miles

   Find the minimum and maximum possible distances between Junctions 18 and 21.

10. (a) The radius of a circle is 4.2 cm. Find the possible range of values for its area and cicumference.

    (b) The circumference of a circle is 8.23 cm. Find the range of possible values for its area.

    (c) The area of a circle is 18.4 cm^2. Find the range of possible values for its radius.

11. (a) The mass of a plant on kitchen scales is 2.474 kg. What is the possible mass of the plant?

    (b) The mass of another plant on scientific scales is 1.6280 kg. What are the upper and lower bounds of the mass of the plant?

(SEG)

12. Twenty video recorders are packed in a single container. Each recorder weighs 12.6 kg, to the nearest 0.1 kg.

    (a) Calculate the lower bound for the total weight of the videos.

    (b) Calculate the difference between the upper and lower bounds for the total weight of the videos.

(SEG)

13. Jafar has a piece of wood that has a length of 30 cm, correct to the nearest centimetre.

    (a) Write down the minimum length of the piece of wood.

    Fatima has a different piece of wood that has a length of 18.4 cm, correct to the nearest millimetre.

    (b) Write down the maximum and minimum lengths between which the length of the piece of wood must lie.

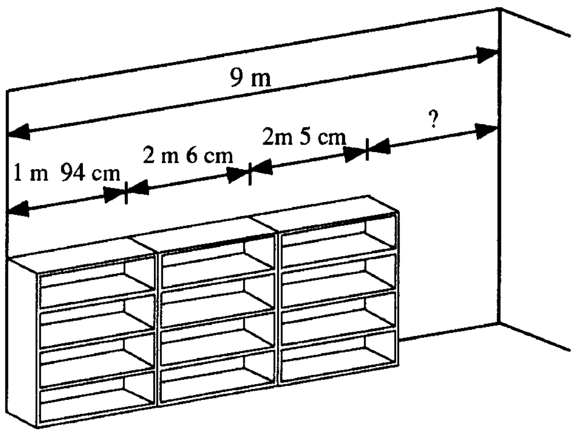
(LON)

14. A full jar of coffee weighs 750 g. The empty jar weighs 545 g. Both weights are accurate to the nearest 5 g.

    Calculate the maximum and minimum possible values of the weight of coffee in the jar.

(MEG)

15. Four shelf units are to be fitted along a library wall. The wall is 9 m long. All measurements are to the nearest cm.



    What is the maximum length of the fourth shelf unit?

(SEG)

16. The length of each side of a square, correct to 2 significant figures, is 3.7 cm.

    (a) Write down the least possible length of each side.

    (b) Calculate the greatest and least possible perimeters of this square.

    (c) (i) When calculating the perimeter of the square, how many significant figures is it appropriate to give in the answer?

    (ii) Explain your answer.

    (d) If this question had referred to a rectangular octagon instead of a square, would your answer to part (c) (i) have been the same?

    Explain your answer.

(LON)

17. David travels from Manchester to London in $3\frac{1}{2}$ hours, measured to the nearest half hour.

    The distance from Manchester to London is 200 miles, measured to the nearest 10 miles.

    (a) Complete these two inequalities:

    $$3.25 \text{ hours} < \text{David's time} < \ ?$$

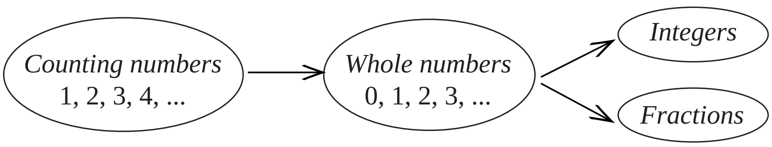
    $$? < \text{Distance from Manchester to London} < \ ?$$

    (b) Calculate upper and lower bounds for the average speed of David's journey. Give these bounds correct to 3 significant figures where appropriate.

(MEG)

6.8 Number System

The number system is classified into various categories.



Integers

These are the set of positive and negative whole numbers, e.g. 1, 2, 3, 364, – 2.

Rational Numbers

A *rational* number is a number which can be written in the form

$$\frac{m}{n}, \quad \text{where } m \text{ and } n \text{ are integers.}$$

For example, $\frac{4}{5}$ is a rational number. A rational number is in its simplest form if m and n have no common factor and n is positive.

Irrational Numbers

There are numbers which *cannot* be written in the form

$$\frac{m}{n}, \quad \text{where } m \text{ and } n \text{ are integers.}$$

Examples of irrational numbers are π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$.

Terminating Decimals

These are decimal numbers which stop after a certain number of decimal places.
For example,

$$\frac{7}{8} = 0.875$$

is a terminating decimal because it stops (terminates) after 3 decimal places.

Recurring Decimals

These are decimal numbers which keep repeating a digit or group of digits; for example

$$\frac{137}{259} = 0.528\ 957\ 528\ 957\ 528\ 957\ ....$$

is a recurring decimal. The six digits 528957 repeat in this order. Recurring decimals are written with dots over the first and last digit of the repeating digits, e.g. $0.\dot{5}28\ 95\dot{7}$

Note

All *terminating* and *recurring* decimals can be written in the form $\frac{m}{n}$, so they are *rational* numbers.

Real Numbers

These are made up of all possible *rational* and *irrational* numbers.

Worked Example 1

Classify the following numbers as integers, rational, irrational, recurring decimals, terminating decimals.

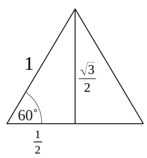
$$\frac{5}{7}, \quad -7, \quad 0.6, \quad 0.4\dot{1}21\dot{3}, \quad \frac{5}{8}, \quad 11, \quad \sqrt{10}, \quad \frac{\pi}{4}, \quad \sqrt{49}, \quad \sin 60^\circ.$$

Solution

Number	Rational	Irrational	Integer	Recurring Decimal	Terminating Decimal
$\frac{5}{7}$	✓			$0.\dot{7}1428\dot{5}$	
-7	✓		✓		
0.6	✓				✓
$0.4\dot{1}21\dot{3}$	✓			✓	
$\frac{5}{8}$	✓				0.625
11	✓		✓		
$\sqrt{10}$		✓			
$\frac{\pi}{4}$		✓			
$\sqrt{49}$	✓		✓		
$\sin 60^\circ$		✓			

Remember that $\sin 60^\circ$ can be found from the sides of an equilateral triangle.

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$



Worked Example 2

Show that the number 0.345 and $0.0\dot{9}1\dot{7}$ are rational.

Solution

For a terminating decimal the proof is straight forward.

$$0.345 = \frac{345}{1000} = \frac{69}{200}$$

which is rational because $m = 69$ and $n = 200$ are integers.

For a recurring decimal, we multipy by a power of ten so that after the decimal point we have only the repeating digits.

$$0.0\dot{9}1\dot{7} \times 10\,000 = 917.917\,917\,917\,...$$

Also
$$0.0\dot{9}1\dot{7} \times 10 = 0.917\,917\,917\,...$$

Subtracting the second equation from the first gives

$$0.0\dot{9}1\dot{7} \times 10\,000 - 0.0\dot{9}1\dot{7} \times 10 = 917$$

$$0.0\dot{9}1\dot{7} \times (10000 - 10) = 917$$

$$0.0\dot{9}1\dot{7} = \frac{917}{9990}$$

which is rational because $m = 917$ and $n = 9990$.

Worked Example 3

Prove that $\sqrt{3}$ is irrational.

Solution

Assume that $\sqrt{3}$ is rational so we can find integers m and n such that $\sqrt{3} = \frac{m}{n}$, and m and n have no common factors. Square both sides.

$$3 = \frac{m^2}{n^2}$$

Multiply both sides by n^2:

$$m^2 = 3n^2$$

Since $3n^2$ is divisible by 3, then m^2 is divisible by 3.

Since m is an integer, m^2 is an integer and 3 is prime, then m must be divisible by 3.

Let $m = 3p$, where p is an integer.

$$3n^2 = m^2 = (3p)^2 = 9p^2$$

$$\Rightarrow \quad n^2 = 3p^2.$$

Since $3p^2$ is divisible by 3, n^2 is divisible by 3 and hence n is divisible by 3.

We have shown that m and n are both divisible by 3.

This contradicts the original assumption that $\sqrt{3} = \dfrac{m}{n}$, where m and n are integers with no factor common. Our original assumption is wrong. $\sqrt{3}$ is *not* rational – it is irrational.

Exercises

1. Classify the following numbers as rational or irrational, terminating or recurring decimals.

   $$\sqrt{100}, \quad 0.\dot{6}, \quad \frac{\pi}{0.2}, \quad \frac{13}{99}, \quad 0.75, \quad \frac{1}{\pi}, \quad \sqrt{11}, \quad \frac{1.6}{4}, \quad \pi^2, \quad \frac{5}{11}, \quad \sin 45°, \quad \cos 60°$$

2. Where possible write each of the following numbers in the form $\dfrac{m}{n}$, where m and n are integers with no common factors

   (a) 0.49 (b) $0.\dot{3}$ (c) $\dfrac{\sqrt{49}}{4}$ (d) $\sqrt{7}$

   (e) 0.417 (f) $0.\dot{1}$ (g) $0.\dot{0}\dot{9}$ (h) $\dfrac{\sqrt{36}}{\sqrt{121}}$

   (i) 0.125 (j) 0.962

3. Write each of the following recurring decimals in the form $\dfrac{m}{n}$, where m and n are integers with no common factors

   (a) $0.\dot{4}\dot{1}$ (b) $0.0\dot{4}0\dot{2}$ (c) $0.\dot{1}4285\dot{7}$ (d) $0.\dot{8}$

   (e) $0.\dot{8}1\dot{2}$ (f) $0.\dot{5}$ (g) $0.\dot{9}0\dot{9}$

4. Given the rational numbers

   $$a = \frac{2}{3}, \quad b = \frac{7}{9}, \quad c = \frac{11}{15}, \quad d = \frac{1}{8},$$

   show that $a+b$, $c+d$, $a+c+d$, ab, cd, bc and abc are rational numbers.

5. Classify each of the following as rational numbers or irrational numbers:

   $$\sin\ 45°, \quad \cos\ 45°, \quad \sin\ 30°, \quad \cos\ 30°, \quad \sin\ 60°, \quad \cos\ 60°.$$

6. If p and q are rational numbers, show that $p+q$ and pq are rational numbers.

   [*Hint*: Let $p = \dfrac{m}{n}$ and $q = \dfrac{r}{s}$, where m, n, r and s are integers.]

7. Prove that the following numbers are irrational:

   $$\sqrt{2}, \quad \sqrt{5}, \quad \sqrt{11}.$$

8. What happens if you try to prove that $\sqrt{4}$ is irrational using the proof by a contradiction method?

9. By choosing some irrational numbers, show that the product or two irrational numbers can be rational or irrational, depending on the numbers chosen.

10. If p and q are irrational numbers, investigate whether $p+q$ can be a rational number.

11. Write down two rational numbers and two irrational numbers which lie in each of the following intervals.

(a) $0<x<1$ (b) $1<x<9$ (c) $-4<x<-2$ (d) $98<x<99$

12. Which of these expressions can be written as recurring decimals and which can be written as non-recurring decimals?

$$\frac{2}{7},\quad \pi,\quad \frac{1}{17},\quad \sqrt{7},\quad \frac{6}{47}$$

(SEG)

13. $$x=\sqrt{a^2+b^2}$$

State whether x is rational or irrational in each of the following cases, and show sufficient working to justify each answer.

(a) $a=5$ and $b=12$ (b) $a=5$ and $b=6$

(c) $a=\sqrt{2}$ and $b=\sqrt{7}$ (d) $a=\frac{3}{7}$ and $b=\frac{4}{7}$

(MEG)

14. (a) Write down **two** irrational numbers that multiply together to give a rational number.

(b) Which of the following numbers are rational?

$$2^{\frac{1}{2}},\quad 2^{-2},\quad 4^{\frac{1}{2}},\quad 4^{-2},\quad \pi^{\frac{1}{2}},\quad \pi^{-2}$$

(SEG)

15. (a) A Mathematics student attempted to define an i*rrational number* as follows:

"An irrational number is a number which, in its decimal form, goes on and on."

(i) Give an example to show that this definition is not correct.

(ii) What must be added to the definition to make it correct?

(b) Which of the following are rational and which are irrational?

$$\sqrt{4\frac{1}{4}},\quad \sqrt{6\frac{1}{4}},\quad \frac{1}{3}+\sqrt{3},\quad \left(\frac{1}{3}\sqrt{3}\right)^2$$

Express each of the rational numbers in the form $\frac{p}{q}$, where p and q are integers.

(MEG)

16. (a) Write down an irrational number that lies between 4 and 5.

(b) N is a rational number which is not equal to zero.

Show clearly why $\frac{1}{N}$ must also be rational.

(NEAB)

17. n is a positive integer such that $\sqrt{n} = 15.4$ correct to 1 decimal place.

(a) (i) Find a value of n.

(ii) Explain why $\sqrt{n}$ is irrational.

(b) Write down a number between 10 and 11 that has a rational square root.

(LON)

6.9 Surds

A *surd* is a number which can be written in the form

$$m + \sqrt{n}$$

where m is any real number and n is any non-negative real number.

$\left(m + \sqrt{n}\right)$ and $\left(m - \sqrt{n}\right)$ are called *conjugate surds*.

$\left(m - \sqrt{n}\right)$ is said to be 'conjugate to' $\left(m + \sqrt{n}\right)$ and vice versa.

A surd is an irrational number. Examples of surds are

$$2 + \sqrt{3}, \quad 1 - \sqrt{7}, \quad \pi + \sqrt{\pi}, \quad \sqrt{2} + \sqrt{3}.$$

Worked Example 1

If $a = 3 + \sqrt{2}$ and $b = 3 - \sqrt{2}$, show that ab is a rational number.

Solution

$$\begin{aligned} ab &= \left(3 + \sqrt{2}\right)\left(3 - \sqrt{2}\right) \\ &= 3 \times 3 - 3 \times \sqrt{2} + \sqrt{2} \times 3 - \sqrt{2} \times \sqrt{2} \\ &= 9 - 0 - 2 \\ &= 7. \end{aligned}$$

$ab = 7$ is rational. (In this case ab is an integer.)

Worked Example 2

If $a = 2 - \sqrt{3}$ and $b = 1 + \sqrt{5}$, simplify $a + b$, $a - b$, ab and $\frac{a}{b}$.

Solution

$$a + b = 2 - \sqrt{3} + 1 + \sqrt{5}$$
$$= 3 + \sqrt{5} - \sqrt{3}$$

$$a - b = 2 - \sqrt{3} - \left(1 + \sqrt{5}\right)$$
$$= 1 - \sqrt{3} - \sqrt{5}$$

$$ab = \left(2 - \sqrt{3}\right)\left(1 + \sqrt{5}\right)$$
$$= 2 + 2\sqrt{5} - \sqrt{3} - \sqrt{15}$$

$$\frac{a}{b} = \frac{\left(2 - \sqrt{3}\right)}{\left(1 + \sqrt{5}\right)} \cdot \frac{\left(1 - \sqrt{5}\right)}{\left(1 - \sqrt{5}\right)}$$
$$= \frac{2 - 2\sqrt{5} - \sqrt{3} + \sqrt{15}}{-4}$$
$$= -\frac{1}{2} + \frac{1}{2}\sqrt{5} + \frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{15}$$

Exercises

1. Expand the following brackets.

(a) $\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)$ (b) $\left(2-\sqrt{3}\right)\left(2+\sqrt{5}\right)$

(c) $\sqrt{3}\left(1-\sqrt{3}\right)$ (d) $\left(7-\sqrt{11}\right)\left(-2+\sqrt{3}\right)$

(e) $\left(4-\sqrt{5}\right)\left(4+\sqrt{5}\right)$ (f) $\left(8-\sqrt{2}\right)\left(4+\sqrt{2}\right)$

(g) $\left(1-\sqrt{17}\right)\left(\sqrt{17}+17\right)$ (h) $\left(2-\sqrt{11}\right)\left(2+\sqrt{11}\right)$

(i) $\left(1-\sqrt{\pi}\right)\left(1+\sqrt{\pi}\right)$ (j) $\left(-3-\sqrt{13}\right)\left(-3+\sqrt{13}\right)$

(k) $\left(-1-\sqrt{5}\right)\left(1-\sqrt{5}\right)$ (l) $\left(\pi+\sqrt{\pi}\right)\left(\pi-\sqrt{\pi}\right)$

(m) $\left(\sqrt{3}+\sqrt{2}\right)\left(1+\sqrt{3}+\sqrt{2}\right)$ (n) $\left(1+\sqrt{2}+\sqrt{5}\right)\left(1-\sqrt{2}-\sqrt{5}\right)$

(o) $\left(2+\sqrt{3}\right)^2$ (p) $\left(1-\sqrt{7}\right)^2$

(q) $\left(-3+\sqrt{5}\right)^2$ (r) $\left(1+\sqrt{2}\right)^3$

2. Write the following without a surd in the denominator.

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{2+\sqrt{5}}{\sqrt{5}}$ (c) $\dfrac{1-\sqrt{3}}{\sqrt{2}}$

(d) $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$ (e) $\dfrac{5+\sqrt{7}}{1+\sqrt{3}}$ (f) $\dfrac{3-\sqrt{2}}{5+\sqrt{3}}$

(g) $\dfrac{1}{\left(1-\sqrt{3}\right)}$ (h) $\dfrac{1}{\left(1+\sqrt{2}\right)^2}$ (i) $\dfrac{7-\sqrt{7}}{7+\sqrt{7}}$

(j) $\dfrac{1+\sqrt{11}}{\sqrt{2}-1}$ (k) $\dfrac{1}{\sqrt{2}-\sqrt{3}}$ (l) $\dfrac{-2+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

3. Show that for any surd, $m+\sqrt{n}$, the product, $p=\left(m+\sqrt{n}\right)\left(m-\sqrt{n}\right)$, is rational. Deduce that if m and n are integers then p is an integer.

4. Show that for any surd, $m+\sqrt{n}$, the quotient, $\dfrac{\left(m+\sqrt{n}\right)}{\left(m-\sqrt{n}\right)}$, is an irrational number.

5. If p and q are different irrational integers, investigate whether

(a) $(p+q)$ (b) pq.

are rational or irrational or could be both, depending on the choice of p and q.

6. (a) Complete the following table, writing YES if true and NO if false.
The first line has been completed for you.

Number	**Rational**	**Irrational**
$\sqrt{2}$	NO	YES
$\dfrac{7}{3}$		
π		
$4\sqrt{2}$		

(b) Write down an irrational number between 6 and 7.

(c) p is a non-zero rational number and q is an irrational number.

(i) Is $p\times q$ irrational or rational?

(ii) Is $p+q$ irrational or rational?

(SEG)

7. (a) (i) Write down a value of x for which $\sqrt{2} \times \sqrt{x}$ is an irrational number.

(ii) Write down a value of y, other than $y = 3$, for which $\sqrt{3} \times \sqrt{y}$ is a rational number.

(b) (i) Write down and add together two irrational numbers which will give an answer that is also an irrational number.

(ii) Write down and add together two irrational numbers which will give an answer that is a rational number.

(MEG)

8. (a) For each of the following state whether it is rational or irrational.

When the result is a rational number, write the answer as a fraction in its simplest form.

(i) $0.272727...$ (ii) $1 + \sqrt{2}$ (iii) $4^0 + 4^{-1} + 4^{-2}$

(b) b is a positive integer greater than 2. $\sqrt{2} \times \sqrt{b}$ is rational.

Write down a possible value for b.

(c) $a^3 = 25,\ b^3 = 27,\ \ c^3 = 0.125,\ \ d^3 = 3\frac{3}{8}$.

Which of the numbers a, b, c, d are irrational?

(SEG)

Investigation

The ancient Egyptians were the first to use fractions. However, they only used fractions with a numerator of one.

Thus they would write $\frac{3}{8}$ *as* $\frac{1}{4} + \frac{1}{8}$, *etc. What do you think the Egyptians would write for the fractions* $\frac{3}{5}$, $\frac{9}{20}$, $\frac{2}{3}$ *and* $\frac{7}{12}$?