17 Quadratic Functions

17.1 Quadratic Expressions

In this section we revisit quadratic formulae and look at the graphs of quadratic functions.

The general formula for a quadratic graph is

$$y = ax^2 + bx + c$$

where *a*, *b* and *c* are constants. We investigate how varying the values of *a*, *b* and *c* changes the graph of the function.

Example 1

(a) Draw the graph $y = x^2$.

(b) Draw the graph $y = x^2 - 1$.

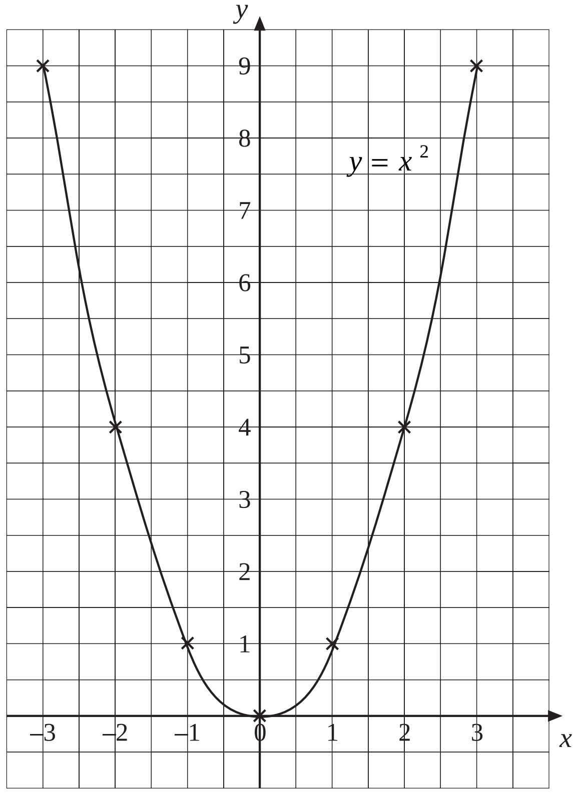
(c) Sketch the graph $y = x^2 + 1$, describing how it relates to the graph $y = x^2$.

Solution

(a) The following table gives a set of values that can be used to draw the graph:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9

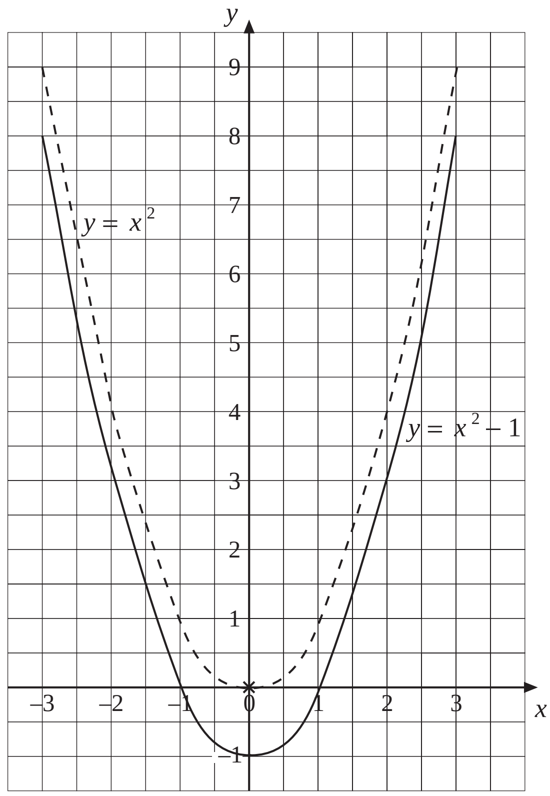
The graph is plotted opposite.



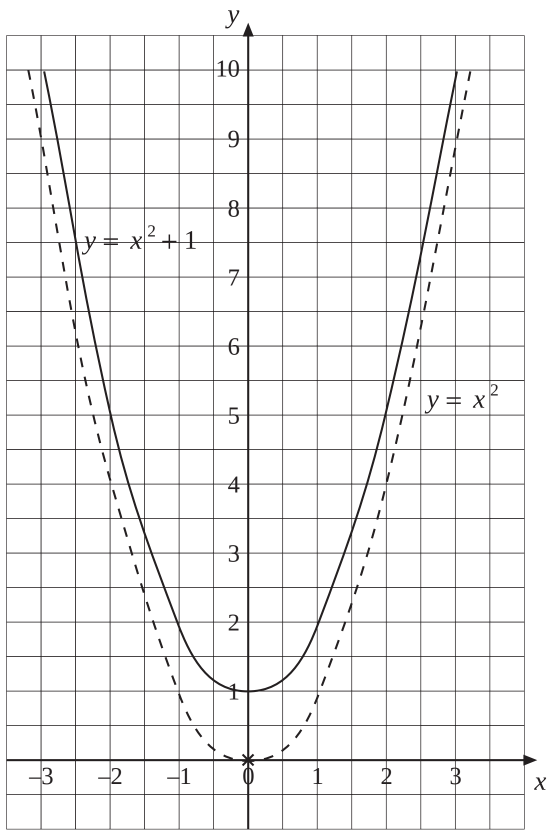
(b) The table gives values for $y = x^2 - 1$.

x	-3	-2	-1	0	1	2	3
x^2	8	3	0	-1	0	3	8

Note that the graph $y = x^2$ is translated *downwards* 1 unit to give the graph $y = x^2 - 1$.



(c) To obtain the graph $y = x^2 + 1$ the graph $y = x^2$ must be translated *upwards* by 1 unit, as shown in this diagram.



Example 2

(a) On the same set of axes, draw the graphs with equations,

$$y = x^2 \quad \text{and} \quad y = \frac{1}{2}x^2.$$

(b) Describe how the two graphs are related.

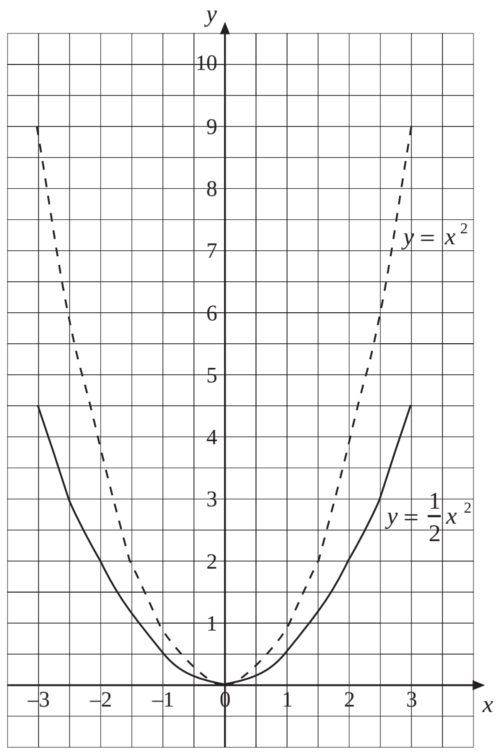
(c) Sketch the graphs $y = \frac{1}{4}x^2$ and $y = 2x^2$.

Solution

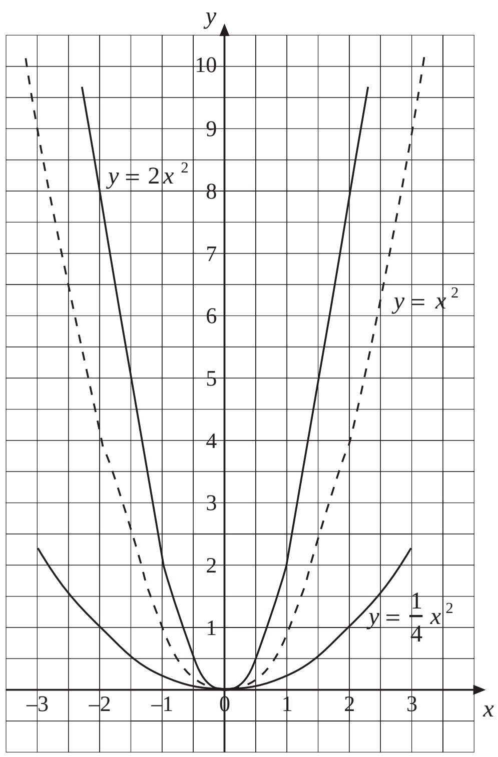
(a) The following table gives the values needed to plot the two graphs:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$\frac{1}{2}x^2$	4.5	2	0.5	0	0.5	2	4.5

(b) The graph $y = \frac{1}{2}x^2$ always has exactly half the height of the graph $y = x^2$, as shown opposite.



(c)



The graph $y = \frac{1}{4}x^2$ will have $\frac{1}{4}$ of the height of the graph $y = x^2$.

The graph $y = 2x^2$ will be twice as high as the graph $y = x^2$.

The two graphs are shown in the diagram on the left.

Exercises

In the following exercises you will explore further how the values of a, b and c change the shape of a quadratic graph.

1. (a) Draw the graph $y = x^2$.

   (b) Draw the graph $y = x^2 + 3$ on the same axes.

   (c) Draw the graph $y = x^2 - 2$ on the same axes.

   (d) Describe how the three graphs are related.

2. (a) Draw the graph $y = -x^2$.

   (b) On the same set of axes, draw the graphs:

   (i) $y = 4 - x^2$ (ii) $y = 9 - x^2$

   (iii) $y = 1 - x^2$ (iv) $y = -1 - x^2$

3. On the same set of axes, sketch the graphs:

   (a) $y = x^2$ (b) $y = 3x^2$

   (c) $y = 4x^2$ (d) $y = \frac{3}{4}x^2$

4. On the same set of axes, sketch the graphs:

   (a) $y = -x^2$ (b) $y = -\frac{1}{2}x^2$

   (c) $y = -\frac{1}{4}x^2$ (d) $y = -\frac{3}{4}x^2$

5. (a) Plot the graphs with equations,

   $y = 2x^2$ and $y = x^2 + 4$.

   (b) What are the coordinates of the points where the two curves intersect?

6. (a) Draw the graphs with equations,

   $y = (x + 1)^2$, $y = (x + 3)^2$ and $y = (x - 2)^2$.

   (b) Describe how each graph is related to the graph $y = x^2$.

   (c) On a new set of axes, sketch the graphs with equations,

   $y = (x - 5)^2$, $y = (x - 3)^2$ and $y = (x + 4)^2$.

7. Sketch the graphs with the following equations:

(a) $y = (x + 1)^2 + 1$ (b) $y = (x - 2)^2 - 3$

(c) $y = (x + 4)^2 - 3$ (d) $y = (x - 3)^2 + 2$

8. (a) Draw the graphs with equations,

$y = x^2 + x$, $y = x^2 + 2x$

$y = x^2 + 4x$, $y = x^2 + 6x$

(b) For each graph, write down the coordinates of the lowest point. What would be the coordinates of the lowest point of the curve
$y = x^2 + bx$?

(c) Draw the graphs of the curves with equations,

$y = x^2 - x$, $y = x^2 - 4x$ and $y = x^2 - 6x$

(d) What would be the coordinates of the lowest point of the curve with equation $y = x^2 - bx$?

9. (a) Draw the graphs with equations,
$y = 2x^2 + 4x + 1$ and $y = 3x^2 + 6x + 2$.

(b) Where does each curve intersect the y-axis?

(c) Where does the curve $y = ax^2 + bx + c$ intersect the y-axis?

(d) Write down the coordinates of the lowest point of each of the curves drawn in part (a).

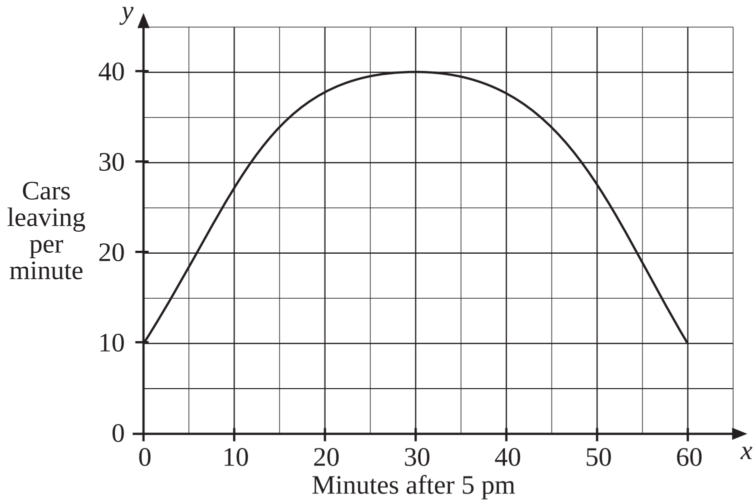
(e) What are the coordinates of the lowest point of the curve
$y = ax^2 + 2ax + c$?

10. (a) Plot the graphs,

$y = x^2 + 5x + 1$, $y = 2x^2 + 8x - 1$ and $y = 3x^2 - 9x + 7$.

(b) What are the coordinates of the lowest point of the curve
$y = ax^2 + bx + c$?

11. The graph shows the rate at which cars left a car park from 5 pm to 6 pm.



The lowest rate was 10 cars per minute at 5 pm and 6 pm.
The highest rate was 40 cars per minute at 5.30 pm.

$y = ax^2 + bx + c$ is the relationship between y, the number of cars leaving per minute, and x, the number of minutes after 5 pm.

(a) Explain how you can work out from the graph that the value of c is 10.

(b) Use the graph to form equations to work out the values of a and b in the equation $y = ax^2 + bx + c$.
Show your working.

(KS3/99/Ma/Ext)

17.2 Quadratic Equations: Factorisation

In this section we look at factorisation and how this can be used to solve quadratic equations. In Unit 11 you factorised expressions; we now take this one stage further to solve equations. In Unit 11 you looked at factorizing expressions with common factors. We now develop this to solving equations with common factors.

Example 1

Factorise: (a) $12x + 8$ (b) $x^3 + x^2$ (c) $3x^2 + 15x$

Solution

(a) $12x + 8 = 4(3x + 2)$

(b) $x^3 + x^2 = x^2(x + 1)$

(c) $3x^2 + 15x = 3x(x + 5)$

Example 2

Factorise: (a) $x^2 + 6x + 8$

(b) $x^2 - 5x + 6$

Solution

As both expressions contain x^2, they will factorise in the form:

$(x \pm \square)(x \pm \square)$

We must determine the missing numbers, and whether a '+' or a '–' sign is required in each bracket.

(a) For $x^2 + 6x + 8$ we require two numbers that multiply together to give 8 and add together to give 6.

So the numbers are 2 and 4.

Hence $x^2 + 6x + 8 = (x + 2)(x + 4)$.

$\times$	x	$+2$
x	x^2	$+2x$
$+4$	$+4x$	$+8$

(b) For $x^2 - 5x + 6$ we require two numbers that multiply together to give 6 and add together to give – 5.

So the numbers are – 2 and – 3.

Hence $x^2 - 5x + 6 = (x - 2)(x - 3)$.

$\times$	x	-2
x	x^2	$-2x$
-3	$-3x$	$+6$

Example 3

Use factorisation to solve the following equations:

(a) $x^2 + 6x = 0$

(b) $x^2 + 3x + 2 = 0$

(c) $x^2 - 8x + 16 = 0$

Solution

(a) First factorise the quadratic expression

$$x^2 + 6x = 0$$

$$x(x + 6) = 0$$

For the left-hand-side to be zero, either:

$x = 0$ or $x + 6 = 0$

The second equation has solution $x = -6$, so the equation $x^2 + 6x = 0$ has solution $x = 0$ or $x = -6$.

(b) First factorise the left-hand-side of the equation:

×	x	$+2$
x	x^2	$+2x$
$+1$	$+1x$	$+2$

$$x^2 + 3x + 2 = 0$$

$$(x+2)(x+1) = 0$$

$x + 2 = 0$ or $x + 1 = 0$

so the equation $x^2 + 3x + 2 = 0$ has solution

$x = -2$ or $x = -1$

(c) Factorise the left-hand-side of the equation:

×	x	-4
x	x^2	$-4x$
-4	$-4x$	$+16$

$$x^2 - 8x + 16 = 0$$

$$(x-4)(x-4) = 0$$

$$(x-4)^2 = 0$$

So $x - 4 = 0$

which means that the equation $x^2 - 8x + 16 = 0$ has just one solution, namely

$x = 4$

Note that in this example the equation has only one solution.

Example 4

Solve the following equations:

(a) $2x^2 = 3x$ (b) $x^2 = 7x - 6$

Solution

(a) First, we rearrange the equation so that it has 0 on the right-hand-side. This first step is essential.

$2x^2 = 3x$

$2x^2 - 3x = 0$ *Subtracting* $3x$ *from both sides*

Now we factorise the left-hand-side and solve the equation as we did in Example 2.

$x(2x - 3) = 0$

so $x = 0$ or $2x - 3 = 0$.

The second equation has solution $x = 1\frac{1}{2}$.

Therefore the equation $2x^2 = 3x$ has solution $x = 0$ or $x = 1\frac{1}{2}$.

(b) Again, the first step is to rearrange the equation so that it has 0 on the right-hand-side.

$$x^2 = 7x - 6$$

$$x^2 - 7x + 6 = 0$$ *Subtracting $7x$ from both sides and adding 6 to both sides*

Now we factorise the left-hand-side and solve the equation.

$$(x-1)(x-6) = 0$$

so $x - 1 = 0$ or $x - 6 = 0$.

Therefore the equation $x^2 = 7x - 6$ has solution $x = 1$ or $x = 6$.

Exercises

1. Factorise the following:

(a)	$3x + 21$	(b)	$5x - 20$	(c)	$x^2 - x$
(d)	$x^2 + 6x$	(e)	$x^3 - x^2$	(f)	$8x + 20x^2$
(g)	$4x - 30x^2$	(h)	$5x + 16x^2$	(i)	$x^4 + x^2$

2. Factorise the following:

(a)	$x^2 + 4x + 3$	(b)	$x^2 - 3x + 2$	(c)	$x^2 - 5x - 14$
(d)	$x^2 - 21x + 20$	(e)	$x^2 + 12x + 35$	(f)	$x^2 - 10x + 25$
(g)	$x^2 - 11x + 30$	(h)	$x^2 - 2x - 63$	(i)	$x^2 - 14x + 48$

3. Solve the following equations:

(a)	$x^2 - 4x = 0$	(b)	$x^2 + 3x = 0$	(c)	$x^2 - 7x = 0$
(d)	$x - 4x^2 = 0$	(e)	$7x - 3x^2 = 0$	(f)	$2x^2 - 5x = 0$

4. Solve the following equations:

(a)	$x^2 - 8x + 12 = 0$	(b)	$x^2 + 2x - 8 = 0$
(c)	$x^2 + x - 6 = 0$	(d)	$x^2 + 3x - 4 = 0$
(e)	$x^2 - 8x + 15 = 0$	(f)	$x^2 - 11x + 18 = 0$
(g)	$x^2 - 6x - 27 = 0$	(h)	$x^2 + 10x + 21 = 0$
(i)	$x^2 - 16x - 17 = 0$	(j)	$x^2 + 17x + 60 = 0$

5. Solve the following equations:

(a) $x^2 = 8x$ (b) $3x^2 = 4x$

(c) $x^2 + 5x = 50$ (d) $x^2 + 70 = 17x$

(e) $x^2 + x = 56$ (f) $x^2 = 14x + 51$

6. (a) Draw the graph of the curve with equation $y = x^2 + 2x - 3$.

(b) Use the graph to explain why the equation $x^2 + 2x - 3 = 0$ has *two* solutions.

7. (a) Draw the graph of the curve with equation $y = x^2 + 2x + 1$.

(b) How many solutions will the equation $x^2 + 2x + 1 = 0$ have?

(c) Check your answer to (b) by factorising $x^2 + 2x + 1$.

8. Use a graph to explain why the equation $x^2 + 2x + 2 = 0$ has no solutions.

9. A rectangle is 3 cm longer than it is wide. The width of the rectangle is *w* cm and the area is 10 cm^2.

(a) Explain why $w(w + 3) = 10$.

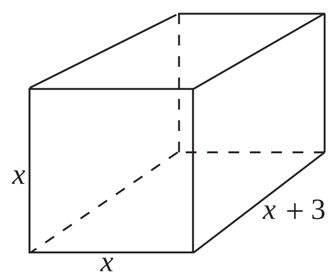
(b) Show that the equation in part (a) can be rewritten in the form $w^2 + 3w - 10 = 0$

(c) Solve the equation $w^2 + 3w - 10 = 0$.

(d) Explain why only one of the solutions to the equation $w^2 + 3w - 10 = 0$ can be applied to the given rectangle.

(e) State the dimensions of the rectangle.

10. The surface area of this cuboid is 18 cm^2. Determine the volume of the cuboid.



11. Solve for y,

$$\frac{9}{y+2} = y + 2$$

17.3 Quadratic Equations: Completing the Square

Completing the square is a useful technique for solving quadratic equations. It is a more powerful technique than factorisation because it can be applied to equations that do not factorise.

When completing the square, an expression like,

$$ax^2 + bx + c \text{ is written in the form } (Ax + B)^2 + C.$$

We will begin with the simple example where $a = 1$. In this case we will write expressions in the form

$$x^2 + bx + c \text{ as } (x + B)^2 + C$$

If we expand $(x + B)^2 + C$ we get $x^2 + 2Bx + B^2 + C$.

Comparing this with $x^2 + bx + c$ shows that

$$b = 2B \quad \text{and} \quad c = B^2 + C$$

which gives $B = \dfrac{b}{2}$ and $C = c - B^2$

Using these two results we can now set about completing the square in some simple cases.

Example 1

Write each of the following expressions in the form $(x + B)^2 + C$.

(a) $x^2 + 6x + 1$ (b) $x^2 + 4x - 2$ (c) $x^2 + 2x$

Solution

(a) Comparing $x^2 + 6x + 1$ with $x^2 + bx + c$ we see that $b = 6$ and $c = 1$ in this case, so

$$B = \frac{b}{2} = \frac{6}{2} = 3 \quad \text{and} \quad C = c - B^2 = 1 - 3^2 = -8.$$

Therefore $x^2 + 6x + 1 = (x + 3)^2 - 8$.

(b) Here $b = 4$ and $c = -2$, so

$$B = \frac{b}{2} = \frac{4}{2} = 2 \quad \text{and} \quad C = c - B^2 = (-2) - 2^2 = -6.$$

Therefore $x^2 + 4x - 2 = (x + 2)^2 - 6$.

(c) Here $B = \dfrac{2}{2} = 1$ and $C = 0 - 1^2 = -1$,

so $x^2 + 2x = (x + 1)^2 - 1$.

Example 2

Solve the following equations by completing the square.

(a) $x^2 - 4x - 5 = 0$ (b) $x^2 + 6x - 1 = 0$

Solution

(a) Completing the square gives,

$$x^2 - 4x - 5 = (x - 2)^2 - 9$$

Now we can solve the equation

$$(x - 2)^2 = 9$$

$$x - 2 = \pm\sqrt{9}$$

$$x - 2 = \pm 3$$

$$x = 2 \pm 3$$

so $x = 5$ or -1

(b) Completing the square gives,

$$x^2 + 6x - 1 = (x + 3)^2 - 10$$

Now we can solve the equation

$$(x + 3)^2 = 10$$

$$x + 3 = \pm\sqrt{10}$$

$$x = -3 \pm \sqrt{10}$$

so $x = 0.162$ or -6.162 to 3 decimal places

Exercises

1. Write each of the following expressions in the form $(x+B)^2+C$.

(a)	x^2+6x	(b)	x^2+4x
(c)	x^2+8x	(d)	x^2-10x
(e)	x^2+7x	(f)	x^2-5x

2. Write each of the following expressions in the form $(x+B)^2+C$:

(a)	x^2+6x+1	(b)	x^2-8x+3
(c)	$x^2+10x-12$	(d)	$x^2+12x+8$
(e)	x^2-4x+1	(f)	x^2-6x-3
(g)	x^2+5x+3	(h)	x^2+3x-4
(i)	x^2+x-2	(j)	x^2-x+3

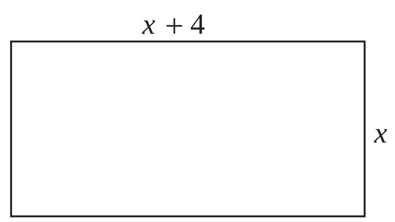
3. Solve each of the following quadratic equation by completing the square:

(a)	$x^2-2x-8=0$	(b)	$x^2+4x+3=0$
(c)	$x^2+8x+12=0$	(d)	$x^2-5x+4=0$
(e)	$x^2-2x-15=0$	(f)	$x^2+3x-28=0$

4. Solve each of the following quadratic equations by completing the square. Give your answers to 2 decimal places.

(a)	$x^2+2x-5=0$	(b)	$x^2+4x-1=0$
(c)	$x^2+6x-5=0$	(d)	$x^2-10x-1=0$
(e)	$x^2+x-3=0$	(f)	$x^2-3x+1=0$
(g)	$x^2+5x-4=0$	(h)	$x^2+3x-5=0$

5. The rectangle shown has an area of 20 cm^2.



(a) Write down an equation for the width x of the rectangle and show that it simplifies to $x^2+4x-20=0$.

(b) Use completing the square to determine the width of the rectangle to 2 decimal places.

6. (a) Write the equation $x^2 - 8x + 18 = 0$ in the form $(x + B)^2 + C = 0$.

   (b) Explain why the equation has no solutions.

7. Simplify each of the following equations and obtain their solutions by completing the square.

   (a) $4x^2 + 20x - 8 = 0$ (b) $20x^2 - 40x + 60 = 0$

   (c) $3x^2 + 6x - 9 = 0$ (d) $5x^2 - 30x - 15 = 0$

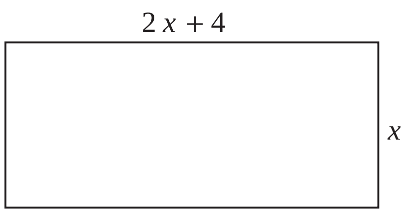
8. The height of a ball at time *t* seconds can be calculated by using the formula

$$h = 20t - 5t^2$$

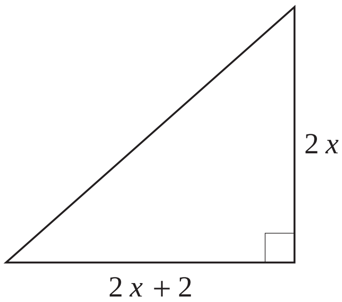
   (a) Calculate the value of h when $t = 2$.

   (b) Determine the values of t for which $h = 15$.

9. The area of the rectangle shown is 30 cm^2.
   Determine the value of x.



10. The area of the triangle shown is 120 cm^2.
    Determine the perimeter of the triangle correct to the nearest millimetre.



11. Use the method of completing the square or the appropriate formula to solve $x^2 + 4x - 2 = 0$.

    Show your working.

    Write your answers showing all the digits on your calculator.

(KS3/95/Ma/Levels 9-10)