15 Trigonometry

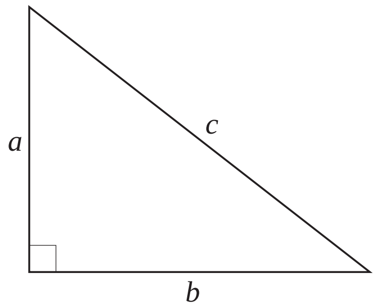
15.1 Pythagoras' Theorem

Pythagoras' Theorem describes the important relationship between the lengths of the sides of a right-angled triangle.

Pythagoras' Theorem

In a right-angled triangle,

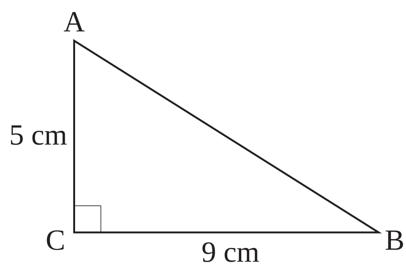
$$a^2 + b^2 = c^2$$



The longest side, c, in a right-angled triangle is called the *hypotenuse*.

Example 1

Calculate the length of the side AB of this triangle:



Solution

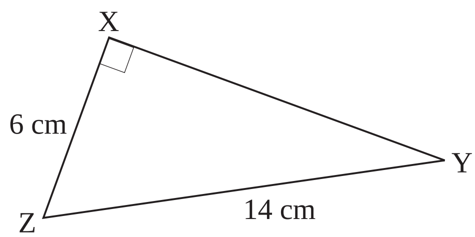
In this triangle,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= 5^2 + 9^2 \\ &= 25 + 81 \\ &= 106 \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{106} = 10.29563014 \text{ cm} \\ &= 10.3 \text{ cm} \quad \text{(to 1 decimal place)} \end{aligned}$$

Example 2

Calculate the length of the side XY of this triangle.



Solution

In this triangle,

$$YZ^2 = XY^2 + XZ^2$$

$$14^2 = XY^2 + 6^2$$

$$196 = XY^2 + 36$$

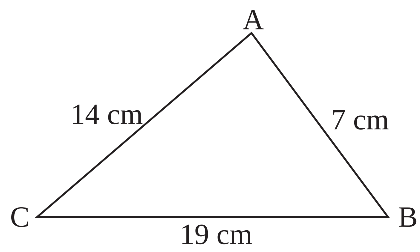
$$XY^2 = 160$$

$$XY = \sqrt{160} = 12.64911064 \text{ cm}$$

$$= 12.6 \text{ cm} \quad \text{(to 1 decimal place)}$$

Example 3

Determine whether or not this triangle contains a right angle.



Solution

If the triangle does contain a right angle, then the longest side, BC, would be the hypotenuse. So, the triangle will be right-angled if $AB^2 + AC^2 = BC^2$.

First consider,

$$AB^2 + AC^2 = 7^2 + 14^2 = 49 + 196$$

$$= 245$$

Now consider,

$$BC^2 = 19^2$$

$$= 361$$

In this triangle,

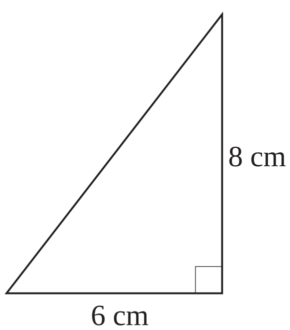
$$AB^2 + AC^2 \neq BC^2$$

so it does *not* contain a right angle.

Exercises

1. Calculate the length of the hypotenuse of each of the triangles shown. Where necessary, give your answers correct to 2 decimal places.

(a)



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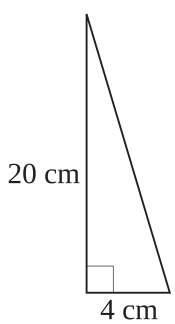

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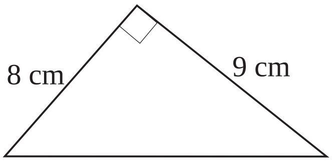

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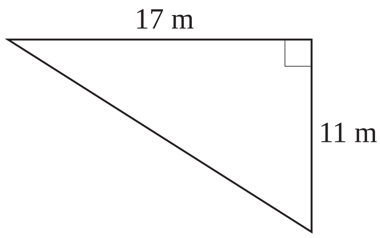
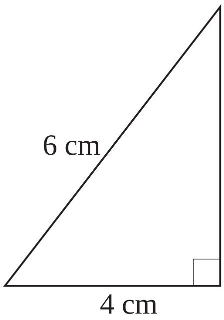


2. Calculate the length of the unmarked side of each of the triangles shown. In each case, give your answer correct to 2 decimal places.

(a)



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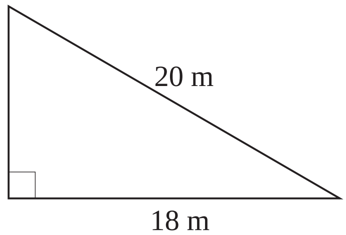

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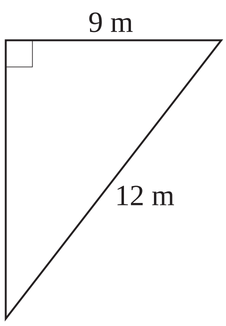

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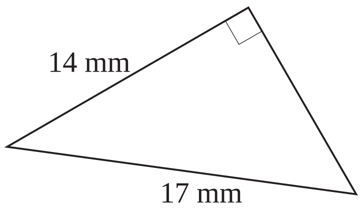
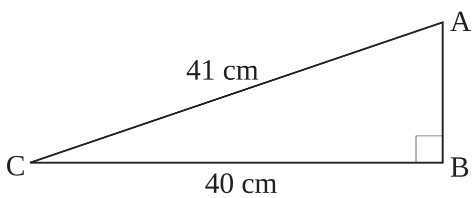
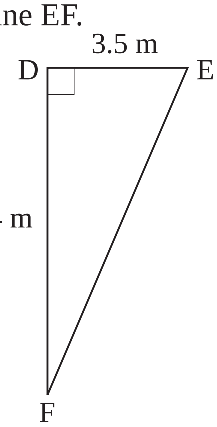


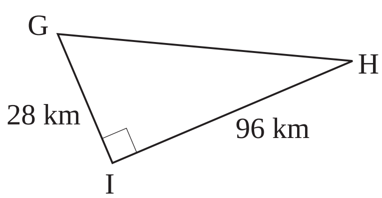
3. (a) Determine AB.



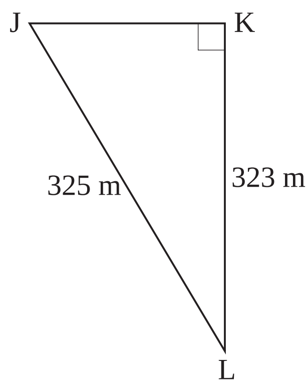
(b) Determine EF.



(c) Determine GH.

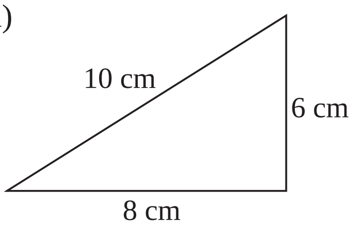


(d) Determine JK.



4. Which of the triangles below contain right angles?

(a)



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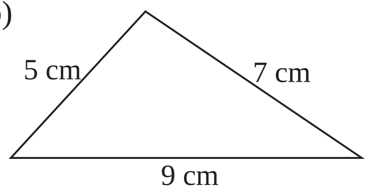

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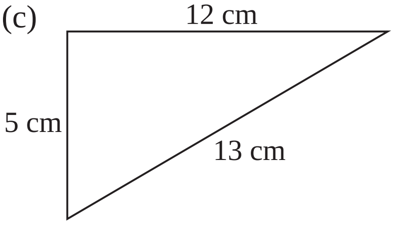

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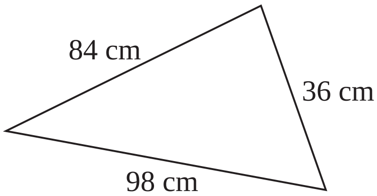
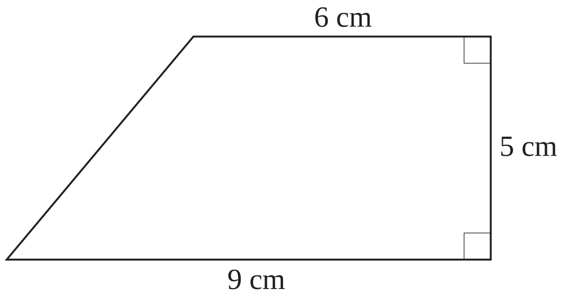


5. Sam walks 100 m north and then 100 m east. How far is she from her starting position? Give your answer to a sensible degree of accuracy.

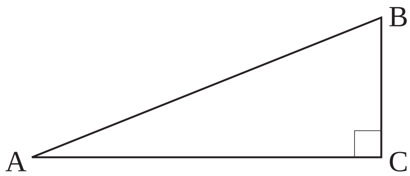
6. Calculate the perimeter of the trapezium shown. Give your answer to the nearest millimetre.



7. The diagram shows a plan for a wheelchair ramp.

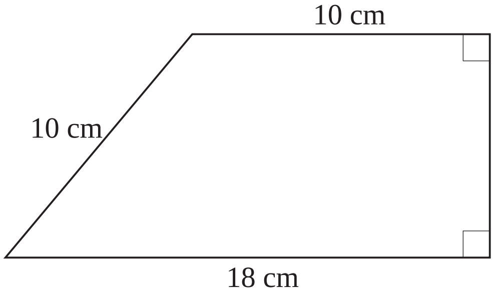
The distance AC is 2 m.

Giving your answer in metres, correct to the nearest cm, calculate the distance AB if:



(a) BC = 20 cm (b) BC = 30 cm

8. Calculate the perimeter and area of this trapezium:



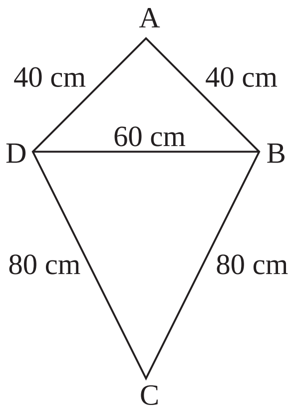
9. A rope is 10 m long. One end is tied to the top of a flagpole. The height of the flagpole is 5 m. The rope is pulled tight with the other end on the ground.

   How far is the end of the rope from the base of the flagpole? Give your answer to a sensible level of accuracy.

10. A ladder leans against a vertical wall. The length of the ladder is 5 m. The foot of the ladder is 2 m from the base of the wall.

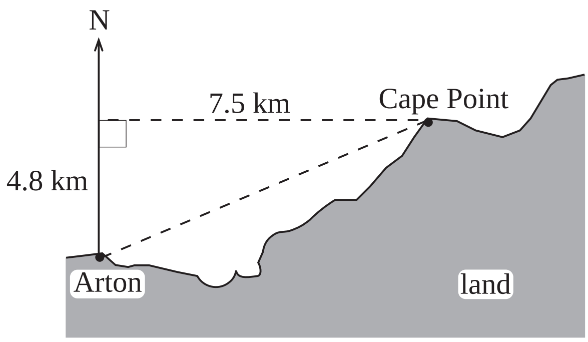
    How high is the top of the ladder above the ground?
    Give your answer to a sensible level of accuracy.

11. Sarah makes a kite from two isosceles triangles, as shown in the diagram.



    Calculate the height, AC, of the kite, giving your answer to the nearest centimetre.

12. *In this question you will get* no marks *if you work out the answer through scale drawing.*



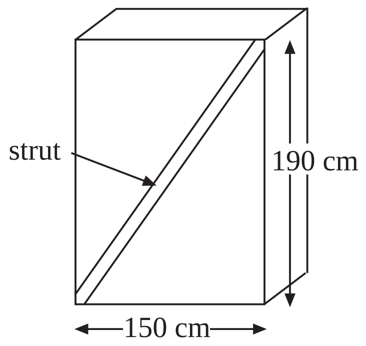
    Cape Point is 7.5 km east and 4.8 km north of Arton.

    Calculate the direct distance from Arton to Cape Point.

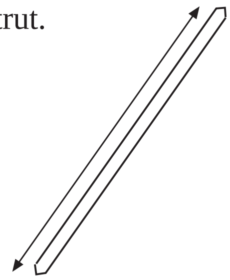
    Show your working.

(KS3/98/Ma/Tier 5-7/P2)

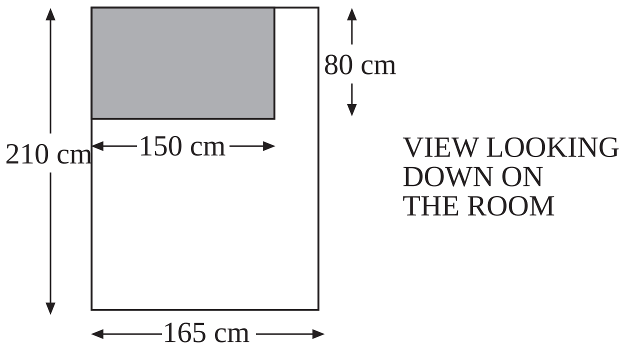
13. A cupboard needs to be strengthened by putting a strut on the back of it like this.



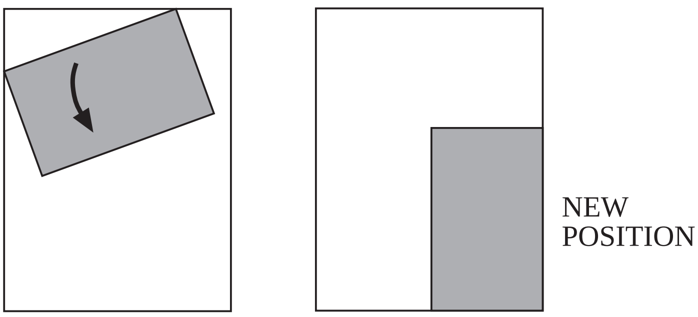
(a) Calculate the length of the diagonal strut.
Show your working.



(b) In a small room the cupboard is in this position.



Calculate if the room is wide enough to turn the cupboard like this and put it in its new position.



Show your working.

(KS3/95/Ma/Levels 5-7/P1)

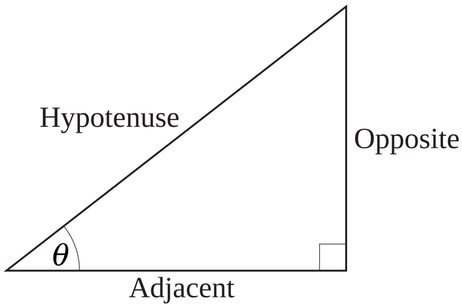
15.2 Trigonometric Functions

In this section we introduce 3 functions: *sine, cosine* and *tangent*, and their use in right-angled triangles. First we look at the conventions used for the names of the sides of a right-angled triangle with respect to one of the angles.

The *adjacent* side is the side joining the angle and the right angle.

The *opposite* side is opposite the angle.

The *hypotenuse* is the side opposite the right angle and is the longest side in the triangle.



Using these definitions, we can write down the trigonometric functions:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{O}}{\text{H}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

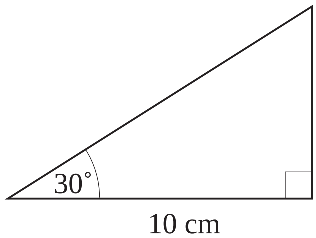
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

Note that we abbreviate sine, cosine and tangent to sin, cos and tan.

In the following Examples and Exercises, we investigate the properties of these trigonometric functions.

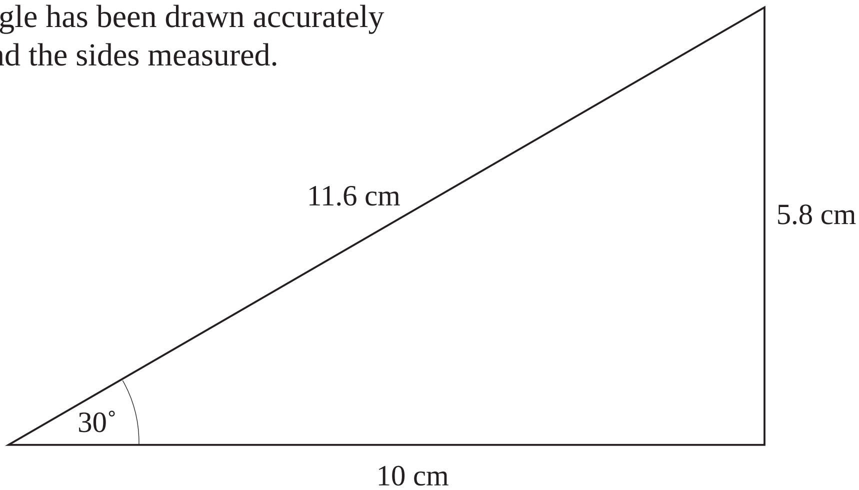
Example 1

Estimate the sin, cos and tan of $30°$, using an accurate drawing of the triangle shown.



Solution

The triangle has been drawn accurately below, and the sides measured.



Here, hypotenuse $= 11.6$ cm, adjacent $= 10$ cm and opposite $= 5.8$ cm, so,

$$\sin 30° = \frac{O}{H} = \frac{5.8}{11.6} = 0.5$$

$$\cos 30° = \frac{A}{H} = \frac{10}{11.6} = 0.86 \text{ (to 2 decimal places)}$$

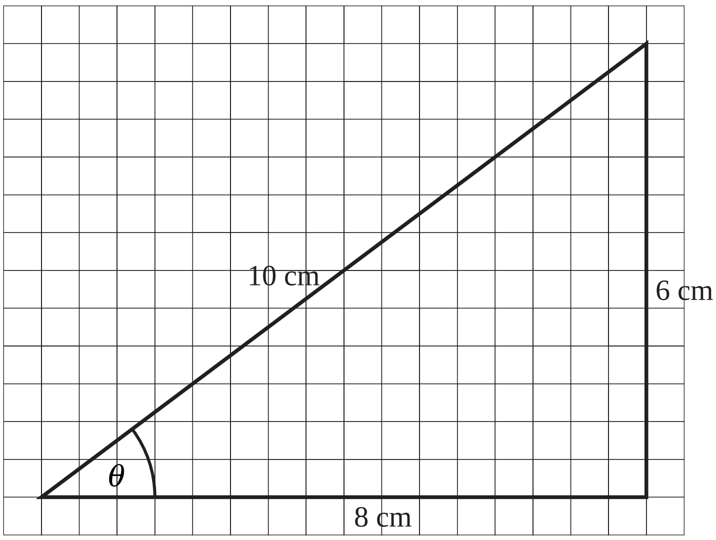
$$\tan 30° = \frac{O}{A} = \frac{5.8}{10} = 0.58$$

Note that if we had drawn a similar right-angled triangle, again containing the $30°$ angle but with different side lengths, then we may have obtained slightly different values for $\sin 30°$, $\cos 30°$ and $\tan 30°$. You can obtain more accurate values of $\sin 30°$, $\cos 30°$ and $\tan 30°$ by using a scientific calculator. If you have a calculator with the trigonometric functions, do this and compare them with the values above.

WARNING: When you use a scientific calculator, always check that it is dealing with angles *in degree mode*.

Example 2

(a) Measure the angle marked in the following triangle:



(b) Calculate the sine, cosine and tangent of this angle.

Solution

(a) In this case the angle can be measured with a protractor as $37°$.

(b) Here we have

$$\begin{aligned} \text{opposite} &= 6 \text{ cm} \\ \text{adjacent} &= 8 \text{ cm} \\ \text{hypotenuse} &= 10 \text{ cm} \end{aligned}$$

$$\sin\theta = \frac{O}{H} = \frac{6}{10} = 0.6 \qquad \cos\theta = \frac{A}{H} = \frac{8}{10} = 0.8 \qquad \tan\theta = \frac{O}{A} = \frac{6}{8} = 0.75$$

Exercises

1. (a) Draw 3 different right-angled triangles that each contain a $60°$ angle.
   (b) Use each triangle to estimate $\sin 60°$, and check that you get approximately the same value in each case.
   (c) Estimate a value for $\cos 60°$.
   (d) Estimate a value for $\tan 60°$.

2. (a) Draw a right-angled triangle that contains an angle of $50°$.
   (b) Use this triangle to estimate:
   (i) $\cos 50°$, (ii) $\sin 50°$, (iii) $\tan 50°$.

3. (a) Draw a right-angled triangle which contains a $45°$ angle.
   (b) Explain why $\sin 45° = \cos 45°$ and state the value of $\tan 45°$.

4. (a) Copy and complete the following table, giving your values correct to 2 significant figures.
   Draw appropriate right-angled triangles to be able to estimate the values.

Angle	*sine*	*cosine*	*tangent*
10°			
20°			
30°			
40°			
50°			
60°			
70°			
80°			

   (b) Use the sin, cos and tan keys on your calculator to check your values.

5. A pupil states that the sine of an angle is 0.5. What is the angle?

6. If the cosine of an angle is 0.17, what is the angle? Give the most accurate answer you can obtain from your calculator and then round it to the nearest degree.

7. What are the values of:

(a)	$\cos 0°$	(b)	$\sin 0°$	(c)	$\sin 90°$
(d)	$\cos 90°$	(e)	$\tan 0°$	(f)	$\tan 90°$

8. Use your calculator to obtain the following, correct to 3 significant figures:

(a)	$\sin 82°$	(b)	$\cos 11°$	(c)	$\sin 42°$
(d)	$\tan 80°$	(e)	$\tan 52°$	(f)	$\tan 38°$

9. Use your calculator to obtain the angle θ, correct to 1 decimal place, if:

(a)	$\cos\theta = 0.3$	(b)	$\sin\theta = 0.77$	(c)	$\tan\theta = 1.62$
(d)	$\sin\theta = 0.31$	(e)	$\cos\theta = 0.89$	(f)	$\tan\theta = 11.4$

10. A student calculates that $\cos\theta = 0.8$.

(a) By considering the sides of a suitable right-angled triangle, determine the values of $\sin\theta$ and $\tan\theta$.

(b) Use a calculator to find the angle θ.

(c) Use the angle you found in part (b) to verify your answers to part (a).

15.3 Calculating Sides

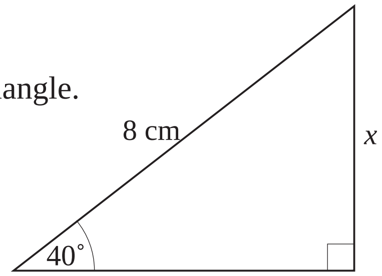
In this section we use the trigonometric functions to calculate the lengths of sides in a right-angled triangle.

Trigonometric Functions

$$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \cos\theta = \frac{\text{A}}{\text{H}} \qquad \tan\theta = \frac{\text{O}}{\text{A}}$$

Example 1

Calculate the length of the side marked x in this triangle.



Solution

In this question we use the *opposite* side and the *hypotenuse*. These two sides appear in the formula for $\sin\theta$, so we begin with,

$$\sin\theta = \frac{\text{O}}{\text{H}}$$

In this case this gives,

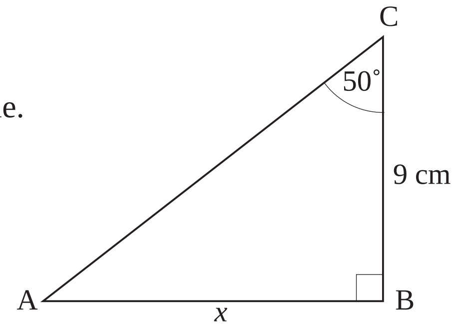
$$\sin 40° = \frac{x}{8}$$

or

$$\begin{aligned} x &= 8 \times \sin 40° \\ &= 5.142300877 \text{ cm} \\ &= 5.1 \text{ cm to 1 decimal place} \end{aligned}$$

Example 2

Calculate the length of the side AB of this triangle.



Solution

In this case, we are concerned with side A B which is the *opposite* side and side BC which is the *adjacent* side, so we use the formula,

$$\tan\theta = \frac{\text{O}}{\text{A}}$$

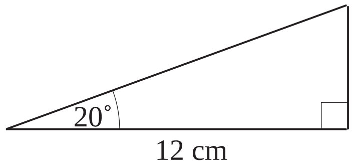
For this problem we have,

$$\tan 50° = \frac{x}{9}$$

$$\begin{aligned} \text{so} \quad x &= 9 \times \tan 50° \\ &= 10.72578233 \text{ cm} \\ &= 10.7 \text{ cm to 1 decimal place} \end{aligned}$$

Example 3

Calculate the length of the hypotenuse of this triangle.



Solution

In this case, we require the formula that links the *adjacent* side and the *hypotenuse*, so we use $\cos\theta$.

Starting with

$$\cos\theta = \frac{O}{H}$$

we can use the values from the triangle to obtain,

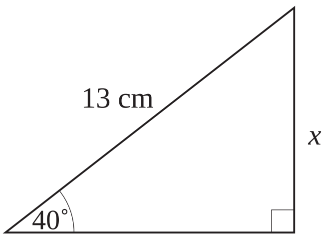
$$\begin{aligned} \cos 20° &= \frac{12}{H} \\ H \times \cos 20° &= 12 \\ H &= \frac{12}{\cos 20°} \\ &= 12.77013327 \text{ cm} \end{aligned}$$

Therefore the hypotenuse has length 12.8 cm to 1 decimal place.

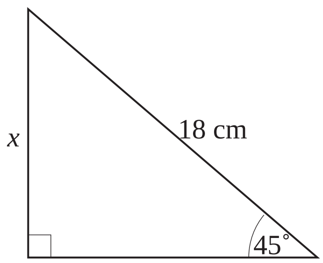
Exercises

1. Use the formula for the sine to determine the length of the side marked x in each of the following triangles. In each case, give your answer correct to 1 decimal place.

(a)



(b)



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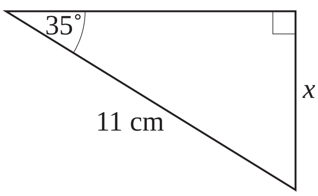

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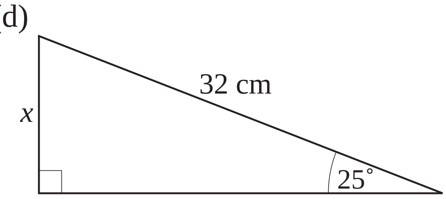
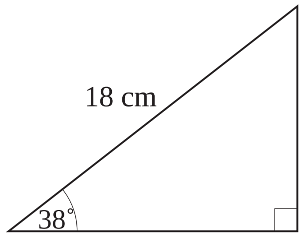
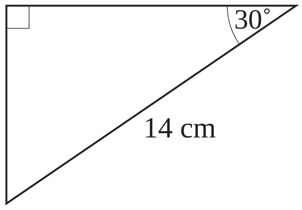


2. Use the formula for the cosine to determine the length of the *adjacent* side in each of the following triangles. Give your answers correct to 1 decimal place.

(a)



(b)



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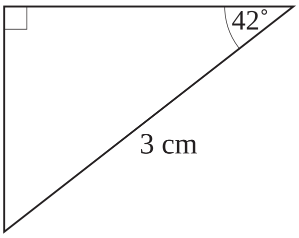

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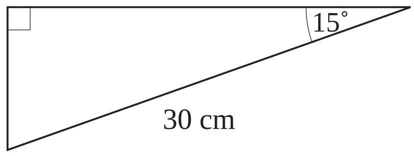
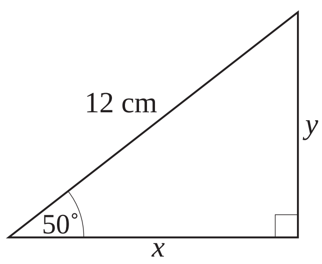
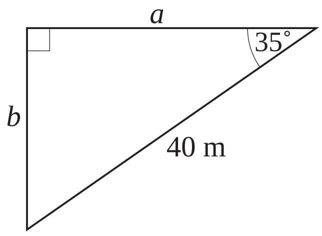


3. Calculate the length of sides indicated by letters in each of the following triangles. Give each of your answers correct to 3 significant figures.

(a)



(b)



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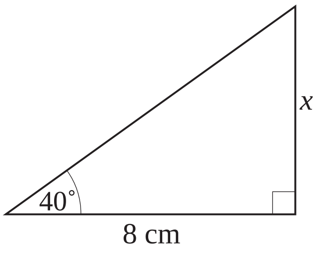
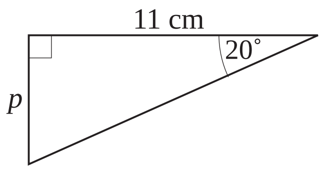


(d)



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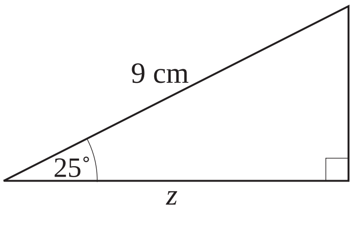
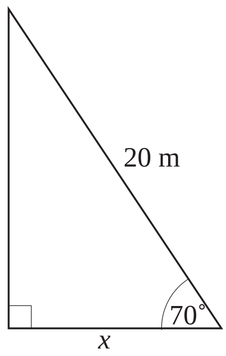
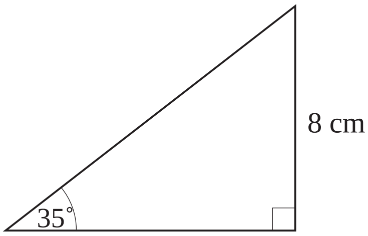


(f)

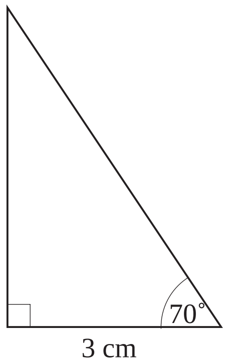


4. Calculate the length of the *hypotenuse* of each of the following triangles. Give each of your answers correct to 3 significant figures.

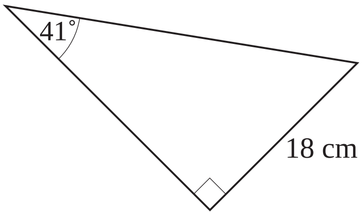
(a)



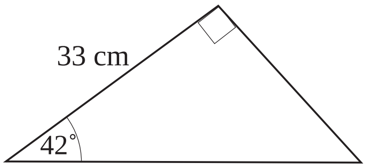
(b)



(c)

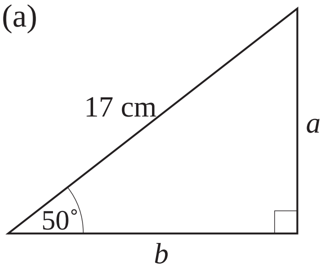


(d)

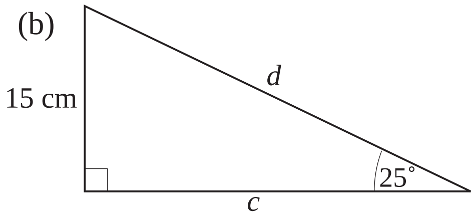


5. Calculate all the lengths marked with letters in the following triangles. Give each of your answers correct to 2 decimal places.

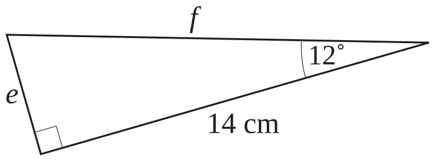
(a)



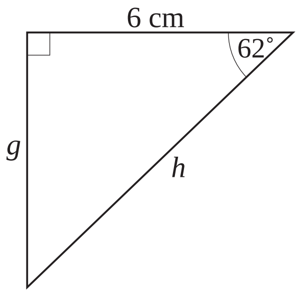
(b)



(c)



(d)



6. A ladder, which has length 6 m, leans against a vertical wall. The angle between the ladder and the horizontal ground is $65°$.

(a) How far is the foot of the ladder from the wall?

(b) What is the height of the top of the ladder above the ground?

In each case, give your answer to the nearest centimetre.

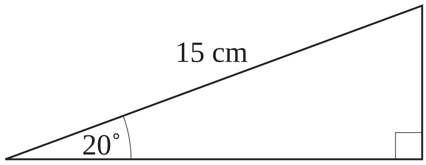
7. A boat sails 50 km on a bearing of $070°$.

(a) How far *east* does the boat travel?

(b) How far *north* does the boat travel?

In each case, give your answer to a sensible level of accuracy.

8. Calculate the *perimeter* and *area* of this triangle. Give your answers correct to 2 decimal places.



9. A ramp has length 6 m and is at an angle of $50°$ above the horizontal. How high is the top of the ramp? Give your answer to a sensible level of accuracy.

10. A rope is stretched from a window in the side of a building to a point on the ground, 6 m from the base of the building. The angle between the rope and the side of the building is $19°$.

    (a) How long is the rope?

    (b) How high is the window?

    In each case, give your answer correct to the nearest centimetre.

15.4 Calculating Angles

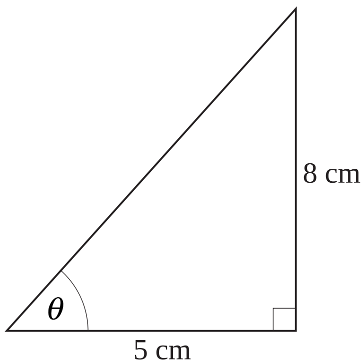
In this section we use trigonometry to determine the sizes of angles in right-angled triangles. On your scientific calculator you will find buttons labelled '$\sin^{-1}$', '$\cos^{-1}$' and '$\tan^{-1}$'. You will need to be able to use these to calculate the angles that will arise in the problems which follow. Again, we start with the three trigonometric functions:

Trigonometric Functions

$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

Example 1

Calculate the angle θ in this triangle.



Solution

In this triangle we are given the lengths of the adjacent and opposite sides, so we will use,

$$\tan\theta = \frac{O}{A}$$

Using the lengths given, we have

$$\begin{aligned} \tan\theta &= \frac{8}{5} \\ &= 1.6 \end{aligned}$$

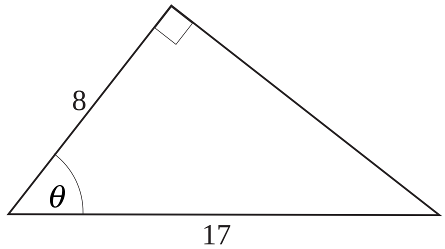
We can then use the $\tan^{-1}$ key on a calculator to obtain

$$\theta = \tan^{-1}(1.6) = 57.99461678°$$

$$= 58.0° \quad \text{(to 1 decimal place)}$$

Example 2

Calculate the angle marked θ in this triangle.



Solution

Because the lengths given are for the adjacent side and the hypotenuse, the formula for $\cos\theta$ must be used.

$$\cos\theta = \frac{A}{H}$$

$$= \frac{8}{17} = 0.470588235$$

$$\theta = \cos^{-1}(0.470588235) = 61.92751306°$$

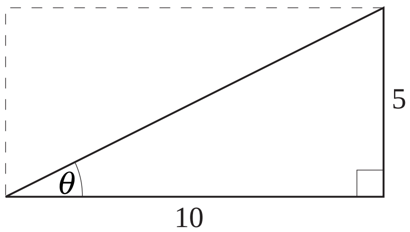
$$= 61.9° \quad \text{(to 1 decimal place)}$$

Example 3

A rectangle has sides of length 5 m and 10 m. Determine the angle between the long side of the rectangle and a diagonal.

Solution

The solution is illustrated in the diagram.



Using the formula for $\tan\theta$ gives

$$\tan\theta = \frac{5}{10}$$

$$= 0.5$$

Then using the $\tan^{-1}$ key on a calculator gives

$$\theta = \tan^{-1}(0.5) = 26.56505118°$$

$$= 26.6° \quad \text{(to 1 decimal place).}$$

Exercises

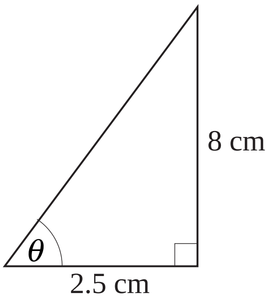
1. Giving your answers, where necessary, correct to 1 decimal place, use your calculator to obtain θ if:

   (a) $\sin\theta = 0.8$ (b) $\cos\theta = 0.5$ (c) $\tan\theta = 1$

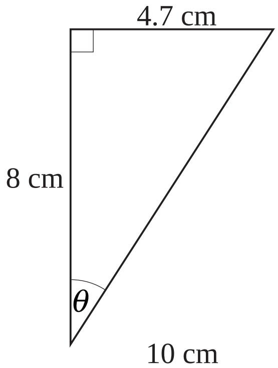
   (d) $\sin\theta = 0.3$ (e) $\cos\theta = 0$ (f) $\tan\theta = 14$

2. Use the tangent function to calculate the angle θ in each of the following diagrams. In each case, give your answer correct to 1 decimal place.

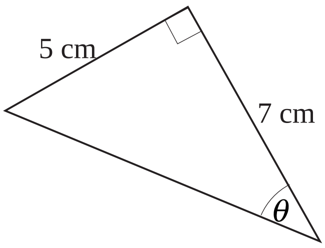
   (a)



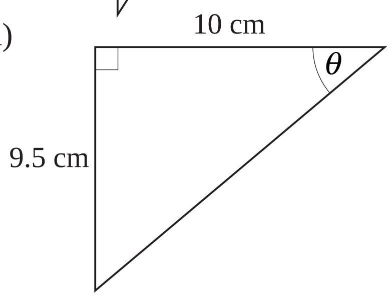
   (b)



   (c)

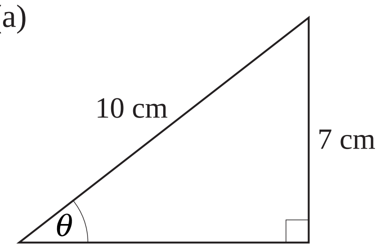


   (d)

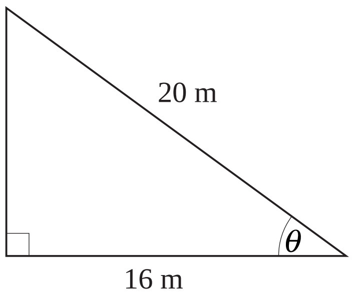


3. Use sine or cosine to calculate the angle θ in each of the following triangles. In each case, give your answer correct to 1 decimal place.

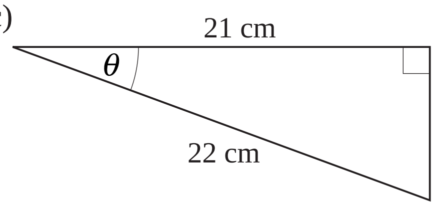
   (a)



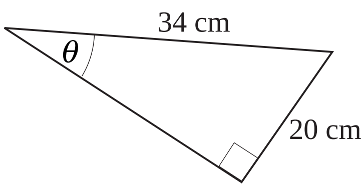
   (b)



   (c)

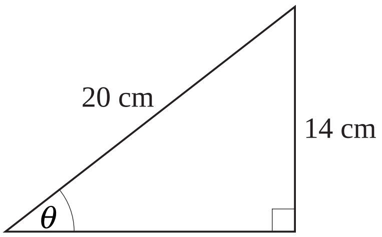


   (d)

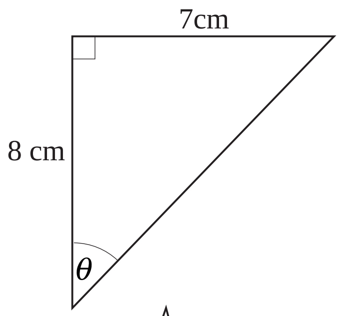


4. Calculate the angle θ in each of the following triangles. In each case, give your answer correct to 1 decimal place.

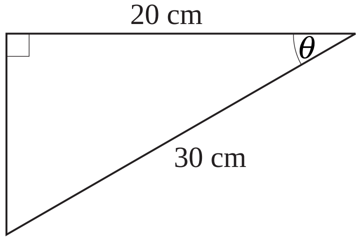
(a)



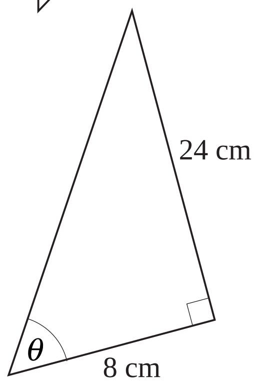
(b)



(c)



(d)

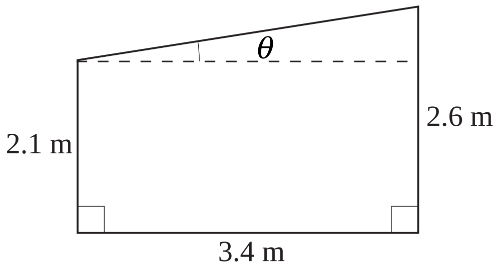


5. A right-angled triangle has sides of length 3 cm, 4 cm and 5 cm.

Determine the sizes of all the angles in the triangle, giving your answers to the nearest degree.

6. The diagram shows the cross-section of a shed.

Calculate the angle θ between the roof and the horizontal. Give your answer to the nearest degree.



7. A ladder of length 6 m leans against a wall. The foot of the ladder is at a distance of 3 m from the base of the wall.

Calculate the angle between the ladder and the ground.

8. A rectangle has sides of length 12 cm and 18 cm.

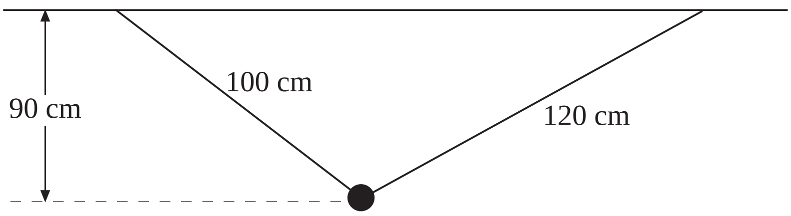
(a) Calculate the length of the diagonal of the rectangle, giving your answer to the nearest millimetre.

(b) Calculate the angle between the diagonal and the shorter side of the rectangle, giving your answer to the nearest degree.

9. As an aeroplane travels 3000 m along a straight flight path, it rises 500 m.

   Calculate the angle between the flight path of the aeroplane and the horizontal. Give your answer to a sensible level of accuracy.

10. A weight hangs from 2 strings as shown in the diagram.

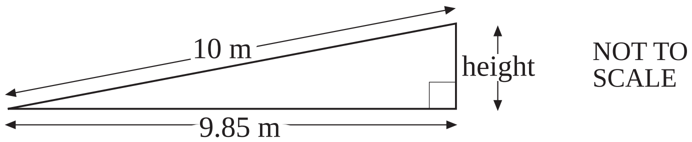


   Calculate the angle between the two strings, giving your answer to the nearest degree.

11. Ramps help people going into buildings.

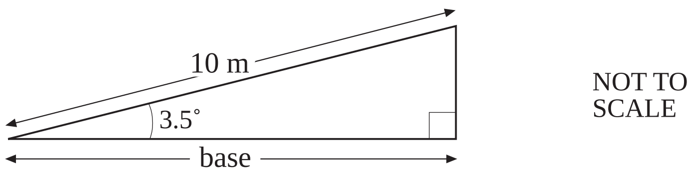
   A ramp that is 10 m long must not have a height greater than 0.83 m.

   (a) Here are the plans for a ramp:



   Is this ramp too high?
   You *must* show calculations to explain your answer.

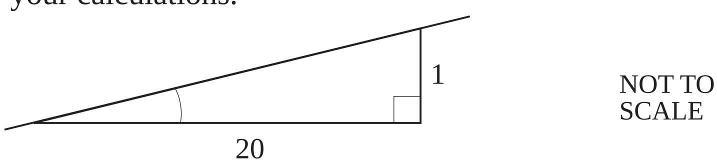
   (b) Here are the plans for a *different* ramp.



   How long is the base of this ramp?
   You *must* show your calculations.

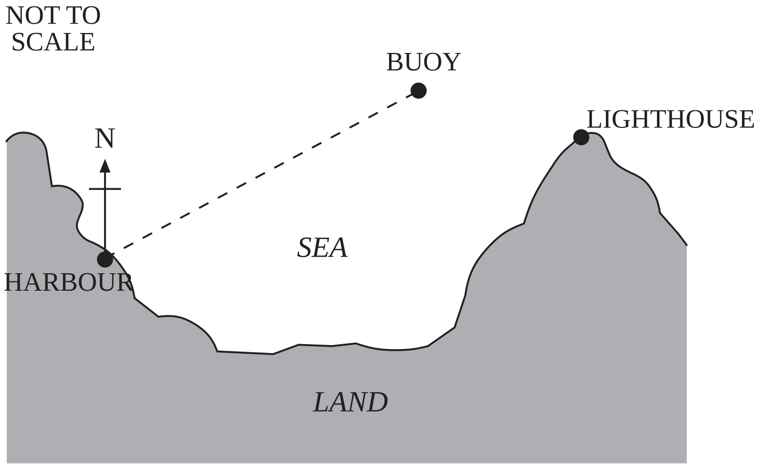
   (c) The recommended gradient of a ramp is 1 in 20.

   What angle gives the recommended gradient?
   You *must* show your calculations.



(KS3/99/Ma/Tier 6-8/P2)

12. A boat sails from the harbour to the buoy.
The buoy is 6 km to the east and 4 km to the north of the harbour.



(a) *Calculate* the shortest distance between the buoy and the harbour.
Give your answer to 1 decimal place.
Show your working.

(b) *Calculate* the bearing of the buoy from the harbour.
Show your working.

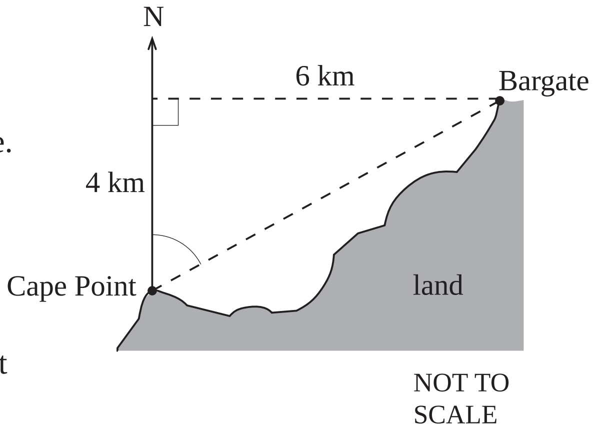
The buoy is 1.2 km to the north of the lighthouse.
The shortest distance from the lighthouse to the buoy is 2.5 km.

(c) *Calculate* how far the buoy is to the west of the lighthouse.
Give your answer to 1 decimal place.
Show your working.

(KS3/96/Ma/Tier 6-8/P1)

13. Bargate is 6 km east and 4 km north of Cape Point.

(a) Steve wants to sail directly from Cape Point to Bargate.
On what bearing should he sail?
Show your working.



(b) Anna sails from Cape Point on a bearing of $048°$.
She stops when she is due north of Bargate.
How far north of Bargate is Anna?
Show your working.

(KS3/98/Ma/Tier 6-8/P2)