1 Base Arithmetic

1.1 Binary Numbers

We normally work with numbers in base 10. In this section we consider numbers in *base 2*, often called *binary numbers*.

In *base 10* we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

In *base 2* we use only the digits 0 and 1.

Binary numbers are at the heart of all computing systems since, in an electrical circuit, 0 represents *no* current flowing whereas 1 represents a current flowing.

In *base 10* we use a system of place values as shown below:

1000	100	10	1		
4	2	1	5	$\rightarrow$	$4 \times 1000 + 2 \times 100 + 1 \times 10 + 5 \times 1$
3	1	0	2	$\rightarrow$	$3 \times 1000 + 1 \times 100 + 2 \times 1$

Note that, to obtain the place value for the next digit to the left, we multiply by 10. If we were to add another digit to the front (left) of the numbers above, that number would represent 10 000s.

In *base 2* we use a system of place values as shown below:

64	32	16	8	4	2	1		
1	0	0	0	0	0	0	$\rightarrow$	$1 \times 64 = 64$
1	0	0	1	0	0	1	$\rightarrow$	$1 \times 64 + 1 \times 8 + 1 \times 1 = 73$

Note that the place values begin with 1 and are multiplied by 2 as you move to the left.

Once you know how the place value system works, you can convert binary numbers to base 10, and vice versa.

Example 1

Convert the following binary numbers to base 10:

(a) 111 (b) 101 (c) 1100110

Solution

For each number, consider the place value of every digit.

(a)

4	2	1		
1	1	1	$\rightarrow$	$4 + 2 + 1 = 7$

The binary number 111 is 7 in base 10.

(b)

4	2	1		
1	0	1	$\rightarrow$	$4 + 1 = 5$

The binary number 101 is 5 in base 10.

(c)

64	32	16	8	4	2	1		
1	1	0	0	1	1	0	$\rightarrow$	$64 + 32 + 4 + 2 = 102$

The binary number 1100110 is 102 in base 10.

Example 2

Convert the following base 10 numbers into binary numbers:

(a) 3 (b) 11 (c) 140

Solution

We need to write these numbers in terms of the binary place value headings 1, 2, 4, 8, 16, 32, 64, 128, ..., etc.

(a)

		2	1
$3 = 2 + 1$	$\rightarrow$	1	1

The base 10 number 3 is written as 11 in base 2.

(b)

		8	4	2	1
$11 = 8 + 2 + 1$	$\rightarrow$	1	0	1	1

The base 10 number 11 is written as 1011 in base 2.

(c)

		128	64	32	16	8	4	2	1
$140 = 128 + 8 + 4$	$\rightarrow$	1	0	0	0	1	1	0	0

The base 10 number 140 is written as 10001100 in base 2.

Exercises

1. Convert the following binary numbers to base 10:

(a) 110 (b) 1111 (c) 1001

(d) 1101 (e) 10001 (f) 11011

(g) 1111111 (h) 1110001 (i) 10101010

(j) 11001101 (k) 111000111 (l) 1100110

2. Convert the following base 10 numbers to binary numbers:

(a) 9	(b) 8	(c) 14
(d) 17	(e) 18	(f) 30
(g) 47	(h) 52	(i) 67
(j) 84	(k) 200	(l) 500

3. Convert the following base 10 numbers to binary numbers:

(a) 5 (b) 9 (c) 17 (d) 33

Describe any pattern that you notice in these binary numbers.
What will be the next base 10 number that will fit this pattern?

4. Convert the following base 10 numbers to binary numbers:

(a) 3 (b) 7 (c) 15 (d) 31

What is the next base 10 number that will continue your binary pattern?

5. A particular binary number has 3 digits.

(a) What are the *largest* and *smallest* possible binary numbers?

(b) Convert these numbers to base 10.

6. When a particular base 10 number is converted it gives a 4-digit binary number. What could the original base 10 number be?

7. A 4-digit binary number has 2 zeros and 2 ones.

(a) List all the possible binary numbers with these digits.

(b) Convert these numbers to base 10.

8. A binary number has 8 digits and is to be converted to base 10.

(a) What is the *largest* possible base 10 answer?

(b) What is the *smallest* possible base 10 answer?

9. The base 10 number 999 is to be converted to binary. How many more digits does the binary number have than the number in base 10?

10. Calculate the difference between the *base 10* number 11111 and the *binary* number 11111, giving your answer in base 10.

1.2 Adding and Subtracting Binary Numbers

It is possible to add and subtract binary numbers in a similar way to base 10 numbers. For example, $1+1+1=3$ in base 10 becomes $1+1+1=11$ in binary. In the same way, $3-1=2$ in base 10 becomes $11-1=10$ in binary. When you add and subtract binary numbers you will need to be careful when 'carrying' or 'borrowing' as these will take place more often.

Key *Addition* Results for Binary Numbers

$$\begin{array}{ccccccc} 1 & + & 0 & = & 1 & & \\ 1 & + & 1 & = & 10 & & \\ 1 & + & 1 & + & 1 & = & 11 \end{array}$$

Key *Subtraction* Results for Binary Numbers

$$\begin{array}{ccccc} 1 & - & 0 & = & 1 \\ 10 & - & 1 & = & 1 \\ 11 & - & 1 & = & 10 \end{array}$$

Example 1

Calculate, using binary numbers:

(a) $111+100$ (b) $101+110$ (c) $1111+111$

Solution

(a)
$$\begin{array}{ccccc} & & 1 & 1 & 1 \\ + & 1 & 0 & 0 & \\ \hline 1 & 0 & 1 & 1 & \\ \hline {\scriptstyle 1} & & & & \end{array}$$

(a)
$$\begin{array}{r} 1\;1\;1 \\ +\,1\;0\;0 \\ \hline 1\;0\;1\;1 \\ \hline {\scriptstyle 1} \end{array}$$

(b)
$$\begin{array}{r} 1\;0\;1 \\ +\,1\;1\;0 \\ \hline 1\;0\;1\;1 \\ \hline {\scriptstyle 1} \end{array}$$

(c)
$$\begin{array}{r} 1\;1\;1\;1 \\ +\quad 1\;1\;1 \\ \hline 1\;0\;1\;1\;0 \\ \hline {\scriptstyle 1\;1\;1} \end{array}$$

Note how important it is to 'carry' correctly.

Example 2

Calculate the binary numbers:

(a) $111-101$ (b) $110-11$ (c) $1100-101$

Solution

(a)
$$\begin{array}{r} 1\;1\;1 \\ -\,1\;0\;1 \\ \hline 1\;0 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 1\;1\;0 \\ -\quad 1\;1 \\ \hline 1\;1 \\ \hline \end{array}$$

(c)
$$\begin{array}{r} 1\;1\;0\;0 \\ -\quad 1\;0\;1 \\ \hline 1\;1\;1 \\ \hline \end{array}$$

Exercises

1. Calculate the binary numbers:

(a) $11+1$ (b) $11+11$ (c) $111+11$

(d) $111+10$ (e) $1110+111$ (f) $1100+110$

(g) $1111+10101$ (h) $1100+11001$ (i) $1011+1101$

(j) $1110+10111$ (k) $1110+1111$ (l) $11111+11101$

2. Calculate the binary numbers:

(a) $11-10$ (b) $110-10$ (c) $1111-110$

(d) $100-10$ (e) $100-11$ (f) $1000-11$

(g) $1101-110$ (h) $11011-110$ (i) $1111-111$

(j) $110101-1010$ (k) $11011-111$ (l) $11110-111$

3. Calculate the binary numbers:

(a) $11+11$ (b) $111+111$

(c) $1111+1111$ (d) $11111+11111$

Describe any patterns that you observe in your answers.

4. Calculate the binary numbers:

(a) $10+10$ (b) $100+100$

(c) $1000+1000$ (d) $10000+10000$

Describe any patterns that you observe in your answers.

5. Solve the following equations, where all numbers, including x, are binary:

(a) $x+11=1101$ (b) $x-10=101$

(c) $x-1101=11011$ (d) $x+1110=10001$

(e) $x+111=11110$ (f) $x-1001=11101$

6. Calculate the binary numbers:

(a) $10-1$ (b) $100-1$

(c) $1000-1$ (d) $10000-1$

Describe any patterns that you observe in your answers.

7. (a) Convert the binary numbers 11101 and 1110 to base 10.
   (b) Add together the two base 10 numbers.
   (c) Add together the two binary numbers.
   (d) Convert your answer to base 10 and compare with your answer to (b).

8. (a) Convert the binary numbers 11101 and 10111 to base 10.
   (b) Calculate the difference between the two base 10 numbers.
   (c) Convert your answer to (b) into a binary number.
   (d) Calculate the difference between the two binary numbers and compare with your answer to (c).

9. Here are 3 binary numbers:

   1110101   1011110   1010011

   Working in binary,
   (a) add together the *two smaller* numbers,
   (b) add together the *two larger* numbers,
   (c) take the smallest number away from the largest number,
   (d) add together all three numbers.

10. Calculate the binary numbers:
    (a) $111 + 101 + 100$
    (b) $11101 + 10011 + 110111$

1.3 Multiplying Binary Numbers

Long multiplication can be carried out with binary numbers and is explored in this section. Note that multiplying by numbers like 10, 100 and 1000 is very similar to working with base 10 numbers.

Example 1

Calculate the binary numbers:

(a) 1011×100 (b) 110110×1000 (c) 11011×10000

Check your answers to (a) and (c) by converting each number to base 10.

Solution

(a) $1011 \times 100 = 101100$

(b) $110110 \times 1000 = 110110000$

(c) $11011 \times 10000 = 110110000$

Checking:

(a)

$$\begin{array}{cccc} 8 & 4 & 2 & 1 \\ \hdashline 1 & 0 & 1 & 1 \end{array} \quad \rightarrow \quad 8+2+1=11$$

$$\begin{array}{ccc} 4 & 2 & 1 \\ \hdashline 1 & 0 & 0 \end{array} \quad \rightarrow \quad 4$$

$$\begin{array}{cccccc} 32 & 16 & 8 & 4 & 2 & 1 \\ \hdashline 1 & 0 & 1 & 1 & 0 & 0 \end{array} \quad \rightarrow \quad 32+8+4=44$$

and $11 \times 4 = 44$, as expected.

(c)

$$\begin{array}{ccccc} 16 & 8 & 4 & 2 & 1 \\ \hdashline 1 & 1 & 0 & 1 & 1 \end{array} \quad \rightarrow \quad 16+8+2+1=27$$

$$\begin{array}{ccccc} 16 & 8 & 4 & 2 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 \end{array} \quad \rightarrow \quad 16$$

$$\begin{array}{ccccccccc} 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 \\ \hdashline 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \quad \rightarrow \quad 256+128+32+16=432$$

and $27 \times 16 = 432$, as expected.

Note: clearly it is more efficient to keep the numbers in binary when doing the calculations.

Example 2

Calculate the binary numbers:

(a) 1011×11 (b) 1110×101

(c) 11011×111 (d) 11011×1001

Solution

(a)

$$\begin{array}{rcccccc} & & & 1 & 0 & 1 & 1 \\ \times & & & & & 1 & 1 \\ \hline & & & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 & 1 & 0 \\ \hline & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \end{array}$$

(b)

$$\begin{array}{rccccccc} & & & & 1 & 1 & 1 & 0 \\ \times & & & & & 1 & 0 & 1 \\ \hline & & & & 1 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & & \end{array}$$

(c)
$$\begin{array}{r} 11011 \\ \times \quad 111 \\ \hline 11011 \\ 110110 \\ 1101100 \\ \hline 10111101 \\ \hline {\scriptstyle 1\ 1\ 1\ 1\ 1\ 1\quad} \end{array}$$

(d)
$$\begin{array}{r} 11011 \\ \times \quad 1001 \\ \hline 11011 \\ 11011000 \\ \hline 11110011 \\ \hline {\scriptstyle 1\ 1\qquad\quad} \end{array}$$

Exercises

1. Calculate the binary numbers:

   (a) 111×10 (b) 1100×100

   (c) 101×1000 (d) 11101×1000

   (e) 11000×10 (f) 10100×1000

   (g) $10100 \div 10$ (h) $1100 \div 100$

   Check your answers by converting to base 10 numbers.

2. Calculate the binary numbers:

   (a) 111×11 (b) 1101×11

   (c) 1101×101 (d) 1111×110

   (e) 11011×1011 (f) 11010×1011

   (g) 10101×101 (h) 10101×111

   (i) 10101×110 (j) 100111×1101

3. Solve the following equations, where all numbers, including x, are binary:

   (a) $\dfrac{x}{11} = 110$ (b) $\dfrac{x}{101} = 101$

   (c) $\dfrac{x}{10} = 111$ (d) $\dfrac{x}{111} = 1011$

4. Multiply each of the following binary numbers by itself:

   (a) 11 (b) 111 (c) 1111

   What do you notice about your answers to parts (a), (b) and (c)?

   What will you get if you multiply 11111 by itself?

5. Multiply each of the following binary numbers by itself:

   (a) 101 (b) 1001 (c) 10001 (d) 100001

   What would you expect to get if you multiplied 1000001 by itself?

6. Calculate the binary numbers:

   (a) $101\,(110 + 1101)$ (b) $1101\,(1111 - 110)$

   (c) $111\,(1000 - 101)$ (d) $1011\,(10001 - 1010)$

7. Here are 3 binary numbers:

   11011 11100 10011

   Working in binary,

   (a) multiply the two *larger* numbers,

   (b) multiply the two *smaller* numbers.

8. (a) Multiply the base 10 numbers 45 and 33.

   (b) Convert your answer to a binary number.

   (c) Convert 45 and 33 to binary numbers.

   (d) Multiply the binary numbers obtained in part (c) and compare this answer with your answer to part (b).

1.4 Other Bases

The ideas that we have considered can be extended to other number bases.

The table lists the digits used in some other number bases.

Base	*Digits Used*
2	0, 1
3	0, 1, 2
4	0, 1, 2, 3
5	0 1, 2, 3, 4

The powers of the base number give the place values when you convert to base 10. For example, for base 3, the place values are the powers of 3, i.e. 1, 3, 9, 27, 81, 243, etc. This is shown in the following example, which also shows how the base 3 number 12100 is equivalent to the base 10 number 144.

Base 3	81	27	9	3	1	
	1	2	1	0	0	$\rightarrow$ $(1 \times 81) + (2 \times 27) + (1 \times 9) + (0 \times 3)$ $+ (0 \times 1) = 144$ in base 10

The following example shows a conversion from base 5 to base 10 using the powers of 5 as place values.

$$\begin{array}{lccccc} \textit{Base 5} & 625 & 125 & 25 & 5 & 1 \\ \hline & 4 & 1 & 0 & 0 & 1 \end{array} \rightarrow (4 \times 625) + (1 \times 125) + (0 \times 25) + (0 \times 5) + (1 \times 1) = 2626 \text{ in base } 10$$

Example 1

Convert each of the following numbers to base 10:

(a) 412 in base 6.

(b) 374 in base 9.

(c) 1432 in base 5.

Solution

(a)
$$\begin{array}{ccc} 36 & 6 & 1 \\ \hline 4 & 1 & 2 \end{array} \rightarrow (4 \times 36) + (1 \times 6) + (2 \times 1) = 152 \text{ in base } 10$$

(b)
$$\begin{array}{ccc} 81 & 9 & 1 \\ \hline 3 & 7 & 4 \end{array} \rightarrow (3 \times 81) + (7 \times 9) + (4 \times 1) = 310 \text{ in base } 10$$

(c)
$$\begin{array}{cccc} 125 & 25 & 5 & 1 \\ \hline 1 & 4 & 3 & 2 \end{array} \rightarrow (1 \times 125) + (4 \times 25) + (3 \times 5) + (2 \times 1) = 242 \text{ in base } 10$$

Example 2

Convert each of the following base 10 numbers to the base stated:

(a) 472 to base 4, (b) 179 to base 7, (c) 342 to base 3.

Solution

(a) For base 4 the place values are 256, 64, 16, 4, 1, and you need to express the number 472 as a linear combination of 256, 64, 16, 4 and 1, but with no multiplier greater than 3.

We begin by writing

$$472 = (1 \times 256) + 216$$

The next stage is to write the remaining 216 as a linear combination of 64, 16, 4 and 1.

We use the fact that

$$216 = (3 \times 64) + 24$$

and, continuing in this way,

$$24 = (1 \times 16) + 8$$
$$8 = (2 \times 4) + 0$$

Putting all these stages together,

$$472 = (1 \times 256) + (3 \times 64) + (1 \times 16) + (2 \times 4) + (0 \times 1)$$
$$= 13120 \text{ in base } 4$$

(b) For base 7 the place values are 49, 7, 1.

$$179 = (3 \times 49) + (4 \times 7) + (4 \times 1)$$
$$= 344 \text{ in base } 7$$

(b) For base 3 the place values are 243, 81, 27, 9, 3, 1.

$$342 = (1 \times 243) + (1 \times 81) + (0 \times 27) + (2 \times 9) + (0 \times 3) + (0 \times 1)$$
$$= 110200 \text{ in base } 3$$

Example 3

Carry out each of the following calculations in the base stated:

(a) $14 + 21$ base 5

(b) $16 + 32$ base 7

(c) $141 + 104$ base 5

(d) $212 + 121$ base 3

Check your answer in (a) by changing to base 10 numbers.

Solution

(a)
$$\begin{array}{r} 1\;4 \\ +\;2\;1 \\ \hline 4\;0 \\ \hline {}_{1}\;\; \end{array}$$

Note that $4 + 1 = 10$ in base 5.

(b)
$$\begin{array}{r} 1\;6 \\ +\;3\;2 \\ \hline 5\;1 \\ \hline {}_{1}\;\; \end{array}$$

Note that $6 + 2 = 11$ in base 7.

(c)
$$\begin{array}{r} 1\;4\;1 \\ +\;1\;0\;4 \\ \hline 3\;0\;0 \\ \hline {\scriptstyle 1\;\;1\;\;} \end{array}$$
Note that $1 + 4 = 10$ in base 5.

(d)
$$\begin{array}{r} 2\;1\;2 \\ +\;1\;2\;1 \\ \hline 1\;1\;1\;0 \\ \hline {\scriptstyle 1\;\;1\;\;1\;\;} \end{array}$$
Note that, in base 3,
$$\begin{aligned} 2 + 1 &= 10 \\ 1 + 2 + 1 &= 11 \\ 2 + 1 + 1 &= 11 \end{aligned}$$

Checking in (a):

(a)
$$\begin{array}{cc} 5 & 1 \\ \hdashline 1 & 4 \end{array} \rightarrow (1 \times 5) + (4 \times 1) = 9$$

$$\begin{array}{cc} 5 & 1 \\ \hdashline 2 & 1 \end{array} \rightarrow (2 \times 5) + (1 \times 1) = 11$$

$$\begin{array}{cc} 5 & 1 \\ \hdashline 4 & 0 \end{array} \rightarrow (4 \times 5) + (0 \times 1) = 20$$

and $9 + 11 = 20$, as expected.

Example 4

Carry out each of the following multiplications in the base stated:

(a) 141×23 in base 5

(b) 122×12 in base 3

(c) 512×24 in base 6

Check your answer to (b) by converting to base 10 numbers.

Solution

(a)
$$\begin{array}{r} 1\;4\;1 \\ \times\quad 2\;3 \\ \hline 1\;0\;2\;3 \\ 3\;3\;2\;0 \\ \hline 4\;3\;4\;3 \\ \hline \end{array}$$
Note that, in base 5,
$$\begin{aligned} 3 \times 4 &= 22 \\ 2 \times 4 &= 13 \end{aligned}$$

(b)

$$\begin{array}{r} 1\;2\;2 \\ \times\quad 1\;2 \\ \hline 1\;0\;2\;1 \\ 1\;2\;2\;0 \\ \hline 1\;0\;0\;1\;1 \\ \hline {\scriptstyle 1\;\;1\;\;1\;\;\;\;\;} \end{array}$$

Note that, in base 3,

$2 \times 2 = 11$

(c)

$$\begin{array}{r} 5\;1\;2 \\ \times\quad 2\;4 \\ \hline 3\;2\;5\;2 \\ 1\;4\;2\;4\;0 \\ \hline 2\;1\;5\;3\;2 \\ \hline {\scriptstyle 1\;\;\;\;1\;\;\;\;\;} \end{array}$$

Note that, in base 6,

$2 \times 4 = 12$

$4 \times 5 = 32$

$2 \times 5 = 14$

Checking in (b):

(b)

9	3	1
1	2	2

$\rightarrow \quad (1 \times 9) + (2 \times 3) + (2 \times 1) = 17$

3	1
1	2

$\rightarrow \quad (1 \times 3) + (2 \times 1) = 5$

81	27	9	3	1
1	0	0	1	1

$\rightarrow \quad (1 \times 81) + (0 \times 27) + (0 \times 9) + (3 \times 1) + (1 \times 1) = 85$

and $17 \times 5 = 85$, as expected.

Exercises

1. Convert the following numbers from the base stated to base 10:

(a) 412 base 5 (b) 333 base 4

(c) 728 base 9 (d) 1210 base 3

(e) 1471 base 8 (f) 612 base 7

(g) 351 base 6 (h) 111 base 3

2. Convert the following numbers from base 10 to the base stated:

(a) 24 to base 3 (b) 16 to base 4

(c) 321 to base 5 (d) 113 to base 6

(e) 314 to base 7 (f) 84 to base 9

(g) 142 to base 3 (h) 617 to base 5

3. Carry out the following additions in the base stated:

(a) $3+2$ in base 4 (b) $5+8$ in base 9

(c) $4+6$ in base 8 (d) $2+2$ in base 3

(e) $6+7$ in base 9 (f) $3+4$ in base 6

4. In what number bases could each of the following numbers be written:

(a) 123 (b) 112 (c) 184

5. Carry out each of the following calculations in the base stated:

(a) $13+23$ in base 4 (b) $120+314$ in base 5

(c) $222+102$ in base 3 (d) $310+132$ in base 4

(e) $624+136$ in base 7 (f) $211+142$ in base 5

(g) $333+323$ in base 4 (h) $141+424$ in base 5

Check your answers to parts (a), (c) and (e) by converting to base 10 numbers.

6. Carry out each of the following multiplications in the base stated:

(a) 3×2 in base 4 (b) 4×3 in base 5

(c) 4×2 in base 6 (d) 3×5 in base 6

(e) 2×2 in base 3 (f) 8×8 in base 9

7. Carry out each of the following multiplications in the base stated:

(a) 121×11 in base 3 (b) 133×12 in base 4

(c) 13×24 in base 5 (d) 142×14 in base 5

(e) 161×24 in base 7 (f) 472×32 in base 8

(g) 414×22 in base 5 (h) 2101×21 in base 3

Check your answers to parts (a), (c) and (e) by converting to base 10 numbers.

8. In which base was each of the following calculations carried out?

(a) $4 + 2 = 11$ (b) $7 + 5 = 13$

(c) $8 \times 2 = 17$ (d) $4 \times 5 = 32$

(e) $11 - 3 = 5$ (f) $22 - 4 = 13$

9. (a) Change 147 in base 8 into a base 3 number.

(b) Change 321 in base 4 into a base 7 number.

(c) Change 172 in base 9 into a base 4 number.

(d) Change 324 in base 5 into a base 6 number.

10. In which base was each of the following calculations carried out?

(a) $171 \times 12 = 2272$ (b) $122 \times 21 = 11102$

(c) $24 \times 32 = 1252$ (d) $333 \times 33 = 23144$